بسم الله الرحمن الرحيم

الحلول المختارة لطلاب الهندسة والعمارة

الجبر الخطي والمعادلات التفاضلية الجزئية

إعداد

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والحاسبات بجامعة أم القرى

الطبعة الثانية

ربيع الآخر ١٤١٤هـ - ١٩٩٣م

الحمد لله رب العالمين والصلاة والسلام على سيد المرسلين

سيدنا محمد وعلى آله وصحابته أجمعين.

وبعد، فهذه مجموعة من المسائل المحلولة في مادة الجبر الخطي والمعادلات

التفاضلية الجزئية من الكتابين المقررين وهما:

1. "Elementary Linear Algebra" by Anton, ed. 4, 1984 &
2. "Elementary Differential Equations" by Rainville & Bedient, ed. 6, 1981.

وقد صدرت الطبعة الأولى في ٢٥/٨/١٤٠٨هـ وهذه الثانية فيها زيادة عليها

تمت بتبويبها وفهرستها بشكل موحد لتسهيل فيها المراجعة.

وقد اكتفيت عن سرد المسألة بذكر رقمها والصفحة التي وردت فيها

في الكتابين بعاليه المقررين للمادة على طلاب كلية الهندسة بجامعة

ام القرى.

والله أسأل أن يفيد بهذا العمل إخواننا الطلاب وأن

يثيب كاتبه إنه جواد كريم.

٢٦/٤/١٤١٠هـ

الفهرس

أولاً : مسائل الجبر الخطي من الكتاب الأول

الصفحة	المسائل المحلولة	الصفحة	المسائل المحلولة
29	$\frac{34}{1X0}$ ، $\frac{38c}{131}$ ، $\frac{8, 9c, 11cd, 12}{139}$	45	$\frac{11}{198}$
30	$\frac{13}{139}$ ، $\frac{8}{144}$	46	$\frac{23}{199}$ $\frac{8}{215}$
31	$\frac{9}{144}$	47	$\frac{9, 10}{216}$
32	$\frac{10}{144}$	48	$\frac{11}{216}$
33	$\frac{14}{144}$ ، $\frac{6a}{152}$	50	$\frac{12}{216}$ ، $\frac{19e}{217}$
34	$\frac{7d}{152}$ ، $\frac{8a, 9}{153}$	51	$\frac{20e}{217}$ ، $\frac{27, 31}{218}$
35	$\frac{9d, 12, 13}{153}$	52	$\frac{1}{219}$ ، $\frac{5}{220}$ ، $\frac{1-3d}{275}$
36	$\frac{3d, 4b, 5, 6b, 7c}{158}$	53	$\frac{5-7acf}{275}$
37	$\frac{8}{158}$ ، $\frac{4, 5}{165}$	54	$\frac{8-10a}{275}$ ، $\frac{18}{276}$
38	$\frac{12, 13c}{165}$ ، $\frac{16}{166}$ ، $\frac{3}{174}$	55	$\frac{1}{283}$ ، $\frac{10}{284}$
39	$\frac{8, 9}{175}$	56	$\frac{12, 14, 18}{284}$
40	$\frac{10b, 11}{175}$ ، $\frac{6}{180}$	57	$\frac{19}{285}$ ، $\frac{19}{290}$
41	$\frac{14}{181}$	58	$\frac{16}{290}$
42	$\frac{16}{181}$ ، $\frac{5}{186}$ ، $\frac{9ace, 11ab}{187}$	59	$\frac{1f}{290}$ ، $\frac{4}{291}$
43	$\frac{14, 15}{187}$ ، $\frac{2b}{197}$	60	$\frac{5, 6}{291}$
44	$\frac{6}{197}$ ، $\frac{9}{198}$	61	$\frac{9, 10}{291}$

الصفحة	المسائل المحلولة	الصفحة	المسائل المحلولة
62	$\frac{7,9}{292}$	72	$\frac{6ac}{318}$
63	$\frac{1}{300}$	73	$\frac{7}{318}$
64	$\frac{2}{300}$	74	$\frac{8}{318}$
65	$\frac{3}{301}$	75	$\frac{9,13ef}{318}$, $\frac{5d}{324}$
66	$\frac{6}{301}$	76	$\frac{6d}{324}$
67	$\frac{2}{307}$	77	$\frac{7}{324}$
68	$\frac{3}{307}$	78	$\frac{8}{324}$
69	$\frac{4}{307}$	79	$\frac{9}{324}$
70	$\frac{5}{307}$	80	$\frac{10}{324}$
71	$\frac{8}{307}$, $\frac{5cdef}{317}$		

ثانياً : مسائل المعادلات التفاضلية الجزئية من الكتاب الثاني

الصفحة	المسائل المحلولة	الصفحة	المسائل المحلولة
81	$\frac{3}{425}$	86	$\frac{8}{473}$
82	$\frac{4}{425}$, $\frac{5,10}{426}$	88	$\frac{1}{480}$
83	427 $\frac{3}{473}$	90	$\frac{2}{480}$
84	$\frac{6}{473}$	91	$\frac{2}{482}$
85	$\frac{7}{473}$	93	$\frac{3}{482}$

الصفحة	المسائل المحلولة
95	$\frac{6}{483}$
96	$\frac{7}{483}$
97	$\frac{4}{486}$
98	$\frac{11}{487}$

$\boxed{\frac{1}{7}}$ Only (b), (d) & (f).

$\boxed{\frac{4b}{7}}$ Augmented matrix $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$\therefore$ The system is:

$$x = 0$$
$$y = 0$$
$$x - y = 1$$

(Note: This system has no solution due to contradiction.)

$\boxed{\frac{9}{8}}$ Augmented Matrix is $\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix} \xrightarrow{R_3 - R_1 - R_2 \to R_3} \begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 0 & 0 & 0 & c-b-a \end{bmatrix}$

$\therefore$ $0x + 0y + 0z = c - b - a$

$\therefore$ For consistancy $c - b - a$ must be zero or $c = b + a$

The condition is $c = a + b$

for $a = 8$, $b = -3$, $c = 5$ check: $c = a + b = 8 - 3 = 5$

$\therefore$ OK Consistant

$\therefore$ $x = b - z = -3 - t$, $z = t$

& $y = a - x - 2z = 8 - (-3 - t) - 2t = 11 - t$

$\therefore$ The solution is $x = -(3 + t)$, $y = 11 - t$ & $z = t$

$\boxed{\frac{3b}{17}}$ Assuming $x_4 = t$ & expanding the third row:

$\therefore x_3 + x_4 = 2 \quad \therefore x_3 = 2 - x_4 = 2 - t$

Second row gives: $x_2 + 0x_3 - x_4 = 4 \quad \therefore x_2 = 4 + x_4 = 4 + t$

First row gives: $x_1 + 3x_4 = 2 \quad \therefore x_1 = 2 - 3x_4 = 2 - 3t$

$\therefore$ The solution to x_1, x_2, x_3, x_4 is respectively

$2-3t, 4+t, 2-t$ & t, where t is a parameter.

$\boxed{\frac{4}{17}}$ (a) $z = 2$, $\therefore y - 2z = -1$ or $y = -1 + 2z = -1 + 4 = 3$

$\therefore x + 2y - 4z = 2 \quad \therefore x = 2 + 4z - 2y = 2 + 8 - 6 = 4$

$\therefore x, y, z$ are respectively $4, 3, 2$.

(b) Let $x_4 = t$

$\therefore x_3 + x_4 = 2 \quad \therefore x_3 = 2 - x_4 = 2 - t$

& $x_2 - 3x_3 - 4x_4 = -2 \quad \therefore x_2 = -2 + 4x_4 + 3x_3 = -2 + 4(t) + 3(2-t) = 4 + t$

& $x_1 + 4x_3 + 7x_4 = 10 \quad \therefore x_1 = 10 - 7x_4 - 4x_3 = 10 - 7t - 4(2-t) = 2 - 3t$

$\therefore x_1, x_2, x_3, x_4$ are respectively $2-3t, 4+t, 2-t, t$; t parameter.

(c) Let $x_5 = u$

$\therefore x_4 + 4x_5 = 2 \quad \therefore x_4 = 2 - 4x_5 = 2 - 4u$

& $x_3 + x_4 + 7x_5 = 3 \quad \therefore x_3 = 3 - 7x_5 - x_4 = 3 - 7(u) - (2 - 4u) = 1 - 3u$

Let $x_2 = v$

$\therefore x_1 + 5x_2 - 4x_3 - 7x_5 = -5 \quad \therefore x_1 = -5 + 7x_5 + 4x_3 - 5x_2 =$

$= -5 + 7u + 4(1-3u) - 5v = -1 - 5u - 5v$

$\therefore x_1, x_2, x_3, x_4$ & x_5 are respectively $-1-5u-5v, v, 1-3u, 2-4u, u$; u & v are parameters

(d) $0x_1 + 0x_2 + 0x_3 = 1$

$\therefore$ The system is inconsistant and hence no solution.

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$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \xrightarrow[3R_1-R_3]{R_2+R_1} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 10 & 2 & 14 \end{bmatrix} \xrightarrow{R_3+10R_2} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & 52 & 104 \end{bmatrix}$$

$\therefore 52x_3 = 104 \quad \therefore x_3 = 2$

& $x_2 = 5x_3 - 9 = 10 - 9 = 1 \quad \therefore x_2 = 1$

& $x_1 = 8 - x_2 - 2x_3 = 8 - 1 - 4 = 3$

$\therefore$ x_1, x_2 & $x_3 = 3, 1$ & 2 respectively.

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$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \xrightarrow{\substack{2R_1-R_2 \\ R_1+R_3 \\ 3R_1-R_4}} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & -3 & 6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & -3 & 6 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{R_2/(-3) \\ R_2+3R_3 \\ R_2-R_4}} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Let $w = t$ & $z = v \quad \therefore y - 2z = 0 \quad \therefore y = 2z = 2v$

& $x - y + 2z - w = -1 \quad \therefore x = y + w - 2z - 1 = 2v + t - 2v - 1 = t - 1$

$\therefore$ x, y, z, w are respectively $t-1, 2v, v, t$; where v & t are parameters.

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$$\begin{bmatrix} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 11 & 7 & 0 & -30 \end{bmatrix} \xrightarrow{\substack{3R_2-5R_1 \\ R_3-R_1 \\ 3R_4-11R_1}} \begin{bmatrix} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & -1 & 11 & 75 \end{bmatrix} \xrightarrow{\substack{-R_2 \\ R_3-R_2 \\ R_4-R_2}} \begin{bmatrix} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_3/7}$$

$$\begin{bmatrix} 3 & 2 & -1 & -15 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_3 = 7$ & $x_2 - 11x_3 = -75 \quad \therefore x_2 = -75 + 77 = 2 \quad \therefore x_1 = -4$

$\therefore$ x_1, x_2 & x_3 are respectively $-4, 2, 7$

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$$\begin{bmatrix} 5 & 2 & 6 \\ -2 & 1 & 3 \end{bmatrix} \xrightarrow{5R_2+2R_1} \begin{bmatrix} 5 & 2 & 6 \\ 0 & 9 & 27 \end{bmatrix} \xrightarrow{R_2/(9)} \begin{bmatrix} 5 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix}$$

$\therefore$ Let $x_3 = t \quad \therefore x_2 + 3x_3 = 0 \quad \therefore x_2 = -3x_3 = -3t$

& $5x_1 + 2x_2 + 6x_3 = 0 \quad \therefore x_1 = \frac{-2x_2 - 6x_3}{5} = \frac{6t - 6t}{5} = 0$

$\therefore$ Besides the trivial solution another solution is x_1, x_2, x_3 respectively equal $0, -3t, t$; where t is a parameter.

$\boxed{\frac{12}{18}}$ $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \xrightarrow[R_3-4R_1]{R_2-3R_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \xrightarrow[R_3-R_2]{-R_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 7 & -14 & 10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$

$\therefore$ The system will have no solution if $a^2-16=0$ & $a-4 \neq 0$ i.e if $a=\pm 4$ & $a \neq 4$, i.e if $a=-4$

& The system will have exactly one solution if $a^2-16 \neq 0$ i.e $a \neq \pm 4$ i.e $a \in \mathbb{R}$ & $a \neq \pm 4$

Finally the system will have infinitely many solutions if $a^2-16=0$ & $a-4=0$ i.e $a=4$

$\boxed{\frac{4}{22}}$ $\begin{bmatrix} 2 & -4 & 1 & 1 \\ 1 & -5 & 2 & 0 \\ 0 & -2 & -2 & -1 \\ 1 & 3 & 0 & 1 \\ 1 & -2 & -1 & 1 \end{bmatrix} \xrightarrow[R_4-R_2,\ R_5-R_4]{2R_2-R_1} \begin{bmatrix} 2 & -4 & 1 & 1 \\ 0 & -6 & 3 & -1 \\ 0 & -2 & -2 & -1 \\ 0 & 8 & -2 & 1 \\ 0 & -5 & -1 & 0 \end{bmatrix} \xrightarrow[3R_3-R_2,\ R_4+4R_3,\ 2R_5-5R_3]{} \begin{bmatrix} 2 & -4 & 1 & 1 \\ 0 & -6 & 3 & -1 \\ 0 & 0 & -9 & -2 \\ 0 & 0 & -10 & -3 \\ 0 & 0 & 8 & 5 \end{bmatrix} \xrightarrow[9R_4-10R_3,\ 10R_5+8R_4]{} \begin{bmatrix} 2 & -4 & 1 & 1 \\ 0 & -6 & 3 & -1 \\ 0 & 0 & -9 & -2 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 26 \end{bmatrix}$

$\therefore$ The system has only the trivial solution

$\boxed{\frac{6}{22}}$ $\begin{bmatrix} \lambda-3 & 1 \\ 1 & \lambda-3 \end{bmatrix} \xrightarrow[R_1-(\lambda-3)R_2]{} \begin{bmatrix} \lambda-3 & 1 \\ 0 & 1-(\lambda-3)^2 \end{bmatrix}$

$\therefore$ For other than trivial solution to exist $\therefore$ $1-(\lambda-3)^2$ must be zero.

$\therefore$ $1-\lambda^2+6\lambda-9=0 \Rightarrow \lambda^2-6\lambda+8=0 \Rightarrow (\lambda-2)(\lambda-4)=0$

$\therefore$ $\lambda = 2$ & 4 gives a nontrivial solution.

$\boxed{\frac{5}{29}}$ (a) $3C-D$ not possible because $2\times 3 \neq 3\times 3$

(b) $(3E)D = \begin{bmatrix} 18 & 3 & 9 \\ -3 & 3 & 6 \\ 12 & 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$

(c) $(AB)C = \left(\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{pmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{pmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(d) $A(BC) = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{pmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(e) $(4B)C+2B$ not possible because $(2\times 2)\times(2\times 3) \neq 2\times 2$

(f) $D+E^2 = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 47 & 10 & 29 \\ 1 & 2 & 5 \\ 35 & 8 & 23 \end{bmatrix} = \begin{bmatrix} 48 & 15 & 31 \\ 0 & 2 & 6 \\ 38 & 10 & 27 \end{bmatrix}$

$\boxed{\frac{16}{37}}$ $AB = \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 10 \\ -1 & 15 \end{bmatrix}$

$\therefore (AB)C = \begin{bmatrix} 14 & 10 \\ -1 & 15 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 40 & 46 \\ 60 & 91 \end{bmatrix} \qquad (1)$

$BC = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 20 & 29 \end{bmatrix}$

$\therefore A(BC) = \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & -4 \\ 20 & 29 \end{bmatrix} = \begin{bmatrix} 40 & 46 \\ 60 & 91 \end{bmatrix} \qquad (2)$

From (1) & (2) $\therefore \underline{(AB)C = A(BC)}$ $\therefore$ OK

$\boxed{\frac{7}{38}}$ $[7A]^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$ $\therefore 7A = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & 2 \\ +4 & 1 \end{bmatrix}$ $\therefore A = \begin{bmatrix} 1 & 2/7 \\ 4/7 & 1/7 \end{bmatrix}$

$\boxed{\frac{8}{38}}$ $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ $\therefore A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}$ & $A^3 = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 26 & 27 \end{bmatrix}$

$\therefore A^{-3} = [A^3]^{-1} = \frac{1}{27}\begin{bmatrix} 27 & 0 \\ -26 & 1 \end{bmatrix}$ & $A^2 - 2A + I = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}$

$\boxed{\frac{9}{38}}$ $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{2R_2 - R_3}$

$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{2R_1 - R_2} \begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{bmatrix}$ $\therefore 2I * A^{-1} = \begin{bmatrix} 1 & -1 & \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$ $\therefore$ A is invertible.

$\boxed{\frac{10}{38}}$ $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$\boxed{\frac{13}{39}}$ $\therefore A^2 - 3A + I = 0$ $\therefore A^2 - 3A = -I$ $\therefore A(A - 3I) = -I$

$\therefore A(3I - A) = I$ $\therefore A^{-1}A(3I - A) = A^{-1}I$ $\therefore \underline{3I - A = A^{-1}}$

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$\boxed{\frac{6c}{46}}$

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3} \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{R_2 + R_3}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right] \xrightarrow[2R_2 - R_3]{2R_1 - R_3} \left[\begin{array}{cccccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right] \begin{array}{l} R_1/2 \Rightarrow \\ R_2/2 \Rightarrow \\ R_3/2 \Rightarrow \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \therefore \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

\# $\therefore$ For the system: $\left.\begin{array}{l} x + z = 18 \\ y + z = 3 \\ x + y = -5 \end{array}\right\}$, the matrix form is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 3 \\ -5 \end{bmatrix}$

or $A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 3 \\ -5 \end{bmatrix} \Rightarrow \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 18 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 18 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \\ 13 \end{bmatrix}$

$\therefore$ The solution is $x = 5$, $y = -10$ & $z = 13$

\# The reduced system is:

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 7 & -14 & 10 \\ 0 & 0 & a^2 - 16 & a - 4 \end{bmatrix}$. For Homogineous it becomes $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -14 \\ 0 & 0 & a^2 - 16 \end{bmatrix}$

Hence, when $a^2 - 16 \neq 0$ we have only trivial solution & when $a^2 - 16 = 0$ we have infinitely many solutions

$\therefore$ Non-trivial solution exists when $a = \pm 4$

$\boxed{\frac{6e}{46}}$

$$\underbrace{\left[\begin{array}{ccc} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right.}_{A}\ \underbrace{\left.\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{I} \xrightarrow[\substack{R_1+R_2 \\ R_1+R_2-R_3}]{} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[\substack{R_2-R_3 \\ R_3/2}]{R_1-\frac{R_3}{2}}$$

$$\underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.}_{I}\ \underbrace{\left.\begin{array}{ccc} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]}_{A^{-1}} \quad \therefore\ A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\# \because\ A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 9 \end{pmatrix} \quad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} -2 \\ -5 \\ 9 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ -8 \end{bmatrix}$$

$$\therefore\ x = 6,\quad y = 9 \ \ \&\ \ z = -8$$

$\boxed{\frac{8}{46}}$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore \left[\begin{array}{ccc|ccc} \cos\theta & \sin\theta & 0 & 1 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\sin\theta \cdot R_1 + \cos\theta R_2]{} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 - \sin\theta \cdot R_2} \begin{bmatrix} \cos\theta & 0 & 0 & \cos^2\theta & -\sin\theta\cos\theta & 0 \\ 0 & 1 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1/\cos\theta} \begin{bmatrix} 1 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 1 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore\ A \text{ is invertible for all } \theta.$$

$\boxed{\frac{13}{47}}$ (a)

$$\left[\begin{array}{cccc|cccc} k_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1/k_1 \\ \to R_2/k_2 \\ R_3/k_3 \\ R_4/k_4 \end{array} \Longrightarrow \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1/k_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/k_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/k_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/k_4 \end{array}\right] \therefore A^{-1} = \begin{bmatrix} 1/k_1 & 0 & 0 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 0 & 0 & 1/k_4 \end{bmatrix}$$

(b)

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1/k_4 \\ R_2/k_3 \\ R_3/k_2 \\ R_4/k_1 \end{array} \Longrightarrow \therefore A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/k_4 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 1/k_1 & 0 & 0 & 0 \end{bmatrix}$$

(c)

$$\left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & k & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1/k \\ (kR_2-R_1)/k^2 \\ \Longrightarrow \\ \Longrightarrow \end{array} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1/k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/k^2 & 1/k & 0 & 0 \\ 0 & 1 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \Longrightarrow \\ (R_3-R_2)/k \\ \Longrightarrow \end{array} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1/k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/k^2 & 1/k & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/k^3 & -1/k^2 & 1/k & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \Longrightarrow \\ \Longrightarrow \\ \Longrightarrow \\ (R_4-R_3)/k \end{array} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1/k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/k^2 & 1/k & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/k^3 & -1/k^2 & 1/k & 0 \\ 0 & 0 & 0 & 1 & -1/k^4 & 1/k^3 & -1/k^2 & 1/k \end{array}\right] \therefore A^{-1} = \begin{bmatrix} 1/k & 0 & 0 & 0 \\ -1/k^2 & 1/k & 0 & 0 \\ 1/k^3 & -1/k^2 & 1/k & 0 \\ -1/k^4 & 1/k^3 & -1/k^2 & 1/k \end{bmatrix}$$

$\boxed{\frac{2}{52}}$

$$\left[\begin{array}{cc|c} 3 & -6 & 8 \\ 2 & 5 & 1 \end{array}\right] \underset{3R_2-2R_1}{\Longrightarrow} \begin{bmatrix} 3 & -6 & 8 \\ 0 & 27 & -13 \end{bmatrix} \therefore x_2 = \frac{-13}{27}, \quad x_1 = \frac{1}{3}\left(8+6\left(\frac{-13}{27}\right)\right) = \frac{8}{3} - \frac{26}{27}$$

$$= \frac{72-26}{27} = \frac{46}{27} \qquad \therefore x_1, x_2 \text{ are respectively } \frac{46}{27}, \frac{-13}{27}.$$

$\boxed{\frac{3}{52}}$ Solution by reduction:

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 1 & 3 & 1 & 4 \\ 1 & 3 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \therefore x_3 = -1 \ \& \ x_2 = 5+(-1) = 4$$

$$\& \ x_1 = -1 - 2(-1) - 2(4) = -7$$

$\therefore x_1, x_2 \ \& \ x_3$ are respectively $-7, 4, -1$

OR:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ \& \ B = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

To find A^{-1} by elimination:

$$\therefore \Big[\underbrace{\begin{matrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{matrix}}_{A} \quad \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\Big] \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2}$$

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_2 + R_3]{R_1 - 2R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2}$$

$$\Big[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} 3 & 2 & -4 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{matrix}}_{A^{-1}}\Big] \qquad \therefore A^{-1} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\therefore AX = B \Rightarrow X = A^{-1}B = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

$\therefore x_1 = -7, \ x_2 = 4 \ \& \ x_3 = -1$ ($\therefore$ OK)

$\boxed{\frac{5}{S2}}$

$$\begin{bmatrix} x+y+z \\ x+y-4z \\ -4x+y+z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} \Rightarrow AX = B$$

$$C_A = \begin{bmatrix} 5 & 15 & 5 \\ 0 & 5 & -5 \\ -5 & 5 & 0 \end{bmatrix} \quad \& \ \det A = 5+15+5 = 25$$

$$\therefore A^{-1} = \frac{C_A^T}{\det A} = \frac{1}{25} \cdot \begin{bmatrix} 5 & 0 & -5 \\ 15 & 5 & 5 \\ 5 & -5 & 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} * B = \frac{1}{5} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 \\ 25 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

$\therefore x = 1 \ , \ y = 5 \ \ \& \ \ z = -1$

$\boxed{\frac{10}{S3}}$ (a) Since diagonal elements are none-zero $\therefore$ invertible

(b) Since one of the diagonal elements is zero $\therefore$ none-invertible.

$\boxed{\frac{2}{S3}}$ $\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \end{bmatrix} \xrightarrow[\cos\theta \cdot R_2 - \sin\theta \cdot R_1]{} \begin{bmatrix} \cos\theta & -\sin\theta & x \\ 0 & 1 & y\cos\theta - x\sin\theta \end{bmatrix}$ $\therefore y' = y\cos\theta - x\sin\theta$ $\& \ x' = \frac{1}{\cos\theta}\left[x + \sin\theta(y\cos\theta - x\sin\theta)\right]$

$= \frac{1}{\cos\theta}\left[y\sin\theta\cos\theta + x(1-\sin^2\theta)\right] = y\sin\theta + x\cos\theta \ \therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$

$\boxed{\frac{5}{S4}}$ Since $x_3 = 2 \ \& \ (a^2-4)x_3 = a-2$

$\therefore$ The system will have no solution when $(a^2-4)2 \neq a-2$

i.e $2a^2 - 8 - a + 2 \neq 0 \Rightarrow 2a^2 - a - 6 \neq 0$

$\therefore (2a+3)(a-2) \neq 0$ i.e $a \neq 2$ or -1.5 or $a \in R - \{a \neq 2, -1.5\}$

& The system will have no unique solution at all for any a

& The system will have many solutions when $(a^2-4)2 = a-2$

i.e. when $a = 2$ or -1.5.

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$\frac{6}{54}$ $\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix} \xrightarrow[R_2-R_1-R_3]{R_2-R_1} \begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & 2-b & 2-b \end{bmatrix}$

$\therefore$ (a) a unique solution will be when $a \cdot a \cdot (2-b) \neq 0$ i.e. $a \neq 0$ & $b \neq 2$

(b) a one parameter solution will be when one equation only is made redundant. This is when $2-b=0$ ie $b=2$ & $a \neq 0$

(c) a two parameter solution will be when two equations are made redundant. This is when $a=0$ & the solution will then be

$\begin{bmatrix} 0 & 0 & b & 2 \\ 0 & 0 & 4-b & 2 \\ 0 & 0 & 2-b & 2-b \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} 0 & 0 & b & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix} \xrightarrow[2R-R_2]{R_2/4} \begin{bmatrix} 0 & 0 & b & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\therefore bz=2$ & $z=1$ $\therefore b=2$

$\therefore$ for $a=0$ & $b=2$ we have a two-parameter solution.

(d) It is seen from (c) that $a=0$ & $b \neq 2$ gives no solution.

$\frac{7}{54}$ $AKB=C \quad \therefore A^TAKB = A^TC \quad \therefore (A^TA)K(BB^T) = A^TCB^T$

$\therefore K(BB^T) = (A^TA)^{-1}A^TCB^T \quad \therefore K = (A^TA)^{-1}A^TCB^T(BB^T)^{-1}$

$A^TA = \begin{bmatrix} 1 & -2 & 1 \\ 4 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -4 & 29 \end{bmatrix} \quad \therefore (A^TA)^{-1} = \frac{1}{158}\begin{bmatrix} 29 & 4 \\ 4 & 6 \end{bmatrix}$

$\therefore (A^TA)^{-1}A^T = \frac{1}{158}\begin{bmatrix} 29 & 4 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 4 & 3 & -2 \end{bmatrix} = \frac{1}{158}\begin{bmatrix} 45 & -46 & 21 \\ 28 & 10 & -8 \end{bmatrix}$

$\therefore (A^TA)^{-1}A^TC = \frac{1}{158}\begin{bmatrix} 45 & -46 & 21 \\ 28 & 10 & -8 \end{bmatrix}\begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix} = \frac{1}{158}\begin{bmatrix} 0 & 316 & -316 \\ 316 & 158 & -158 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & -1 \end{bmatrix}$

$\therefore (A^TA)^{-1}A^TCB^T = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 2 \end{bmatrix}$

$\therefore BB^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad \therefore (BB^T)^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

$\therefore K = (A^TA)^{-1}A^TCB^T(BB^T)^{-1} = \begin{bmatrix} 0 & 4 \\ 4 & 2 \end{bmatrix} \cdot \frac{1}{8}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 0 & 16 \\ 8 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

$\therefore K = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

$\frac{9a}{54}$ $A^4=0 \quad \therefore A^4 - I = -I \quad \therefore I - A^4 = I \quad \therefore (I-A^2)(I+A^2) = I \quad \therefore (I-A)(I+A)(I+A^2) = I$

$\therefore (I-A)(I+A+A^2+A^3) = I \quad \therefore \underline{I+A+A^2+A^3 = (I-A)^{-1}}$

$\frac{9b}{54}$ $A^{n+1} = 0$

$\therefore I - A^{n+1} = I - 0 = I$

$\therefore (I-A)(I+A+A^2+\dots+A^n) = I$

$\therefore (I-A)^{-1} = I + A + A^2 + \dots + A^n$

$\boxed{\frac{10}{54}}$. $y = ax^2 + bx + c$ through $(1,2)$, $(-1,6)$ & $(2,3)$

$\therefore$ $\left.\begin{array}{l} 2 = a + b + c \\ , \; 6 = a - b + c \\ \& \; 3 = 4a + 2b + c \end{array}\right\} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{bmatrix}$ is the augmented matrix

$\xRightarrow[R_3-4R_1]{R_1-R_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & -2 & -3 & -5 \end{bmatrix} \xRightarrow[R_3+R_2]{} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & -3 & -9 \end{bmatrix}$ $\therefore$ $c = \frac{-9}{-3} = 3$, $b = \frac{-4}{2} = -2$ & $a = 2 - c - b = 2 - 3 + 2 = 1$

$\therefore$ $y = x^2 - 2x + 3$ is the required graph.

$\boxed{\frac{14}{55}}$, Multiplying both sides with $(3x-1)(x^2+1)$

$\therefore$ $x^2 + x - 2 = A(x^2+1) + (Bx + C)(3x - 1)$

$= Ax^2 + A + 3Bx^2 + 3Cx - Bx - C$

$= (A+3B)x^2 + (3C - B)x + A - C$

Equating corresponding terms:

$\therefore$ $\begin{array}{l} A + 3B = 1 \\ , \; 3C - B = 1 \\ \& \; A - C = -2 \end{array} \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 3 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ $\therefore$ $\Delta = 1 + 9 = 10$, $\Delta_A = -18 + 1 + 3 = -14$ & $\Delta_B = 3 - 1 + 6 = 8$, $\Delta_C = 2 + 3 + 1 = 6$

$\therefore$ $A = -1.4$, $B = 0.8$ & $C = 0.6$

$\boxed{\frac{10}{62}}$ $\begin{vmatrix} k & -3 & 9 \\ 2 & 4 & k+1 \\ 1 & k^2 & 3 \end{vmatrix} = 12k - 3(k+1) + 18k^2 - 36 + 18 - k^3(k+1) = -k^4 - k^3 + 18k^2 + 9k - 21$

$\boxed{\frac{11}{62}}$ (a) $(\lambda-1)(\lambda-4) + 2 = 0$ $\therefore$ $\lambda^2 - 5\lambda + 4 + 2 = 0$ $\therefore$ $\lambda^2 - 5\lambda + 6 = 0$

$\therefore$ $(\lambda-3)(\lambda-2) = 0$ $\therefore$ $\lambda = 2$ or 3.

(b) $(\lambda-6)(\lambda(\lambda-4)+4) = 0$ $\therefore$ $(\lambda-6)(\lambda^2 - 4\lambda + 4) = 0$ $\therefore$ $(\lambda-6)(\lambda-2)^2 = 0$

$\therefore$ $\lambda = 2$ or 6

$\boxed{\frac{14}{63}}$

$$\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix} = \text{(by expanding through C2)}$$

$$-4\begin{vmatrix} 2 & 6 & 3 \\ 4 & 2 & 5 \\ 1 & -2 & 4 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & -3 & 1 \\ 2 & 6 & 3 \\ 1 & -2 & 4 \end{vmatrix} = -4(16+30-24-6-96+20)$$

$$+1\cdot(24-9-4-6+24+6) = -4*(-60) + 1*(35) = 240+35 = 275$$

∴ The value of the above determinant is 275

\# By reduction:

$$\therefore \begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix} \begin{matrix} \\ R_2-2R_1 \\ R_3-4R_1 \\ R_4-R_1 \end{matrix} \begin{vmatrix} 1 & 4 & -3 & 1 \\ 0 & -8 & 12 & 1 \\ 0 & -17 & 14 & 1 \\ 0 & -4 & 1 & 3 \end{vmatrix} \underset{C_4 \Leftrightarrow C_2}{=} - \begin{vmatrix} 1 & 1 & -3 & 4 \\ 0 & 1 & 12 & -8 \\ 0 & 1 & 14 & -17 \\ 0 & 3 & 1 & -4 \end{vmatrix}$$

$$= - \begin{matrix} \\ \\ R_3-R_2 \\ R_4-3R_2 \end{matrix} \begin{vmatrix} 1 & 1 & -3 & 4 \\ 0 & 1 & 12 & -8 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & -35 & 20 \end{vmatrix} = -\frac{1}{2} * \begin{matrix} \\ \\ \\ 2R_4+35R_3 \end{matrix} \begin{vmatrix} 1 & 1 & -3 & 4 \\ 0 & 1 & 12 & -8 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & -275 \end{vmatrix} =$$

$$= -\frac{1}{2} * 1 * 1 * 2 * (-275) = 275$$

∴ By reduction, the value of determinant is 275 (OK)

$\boxed{\frac{15}{63}}$ ⓐ $\Delta = +1*(-2)*(+3)*(-4)*(+5) = 120$

ⓑ $\Delta = (-4)*(+2)*(-3)*(-1)*(5) = -120$

$\boxed{\frac{4}{68}}$ $$\begin{vmatrix} 1 & -2 & 0 \\ -3 & 5 & 1 \\ 4 & -3 & 2 \end{vmatrix} \begin{matrix} \\ 3R_1+R_2 \\ R_3-4R_1 \end{matrix} \begin{vmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 5 & 2 \end{vmatrix} \begin{matrix} \\ \\ R_3+5R_2 \end{matrix} \begin{vmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{vmatrix} = -7$$

$\boxed{\frac{6}{68}}$

$$\begin{vmatrix} 3 & 6 & 9 & 3 \\ -1 & 0 & 1 & 0 \\ 1 & 3 & 2 & -1 \\ -1 & -2 & -2 & +1 \end{vmatrix} \underset{\substack{R_3+R_2 \\ R_4+R_3}}{\overset{R_1/3}{=\!=\!=}} 3\begin{vmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 1 & 9 \\ 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & 0 \end{vmatrix} \underset{R_3-3R_4}{\overset{R_2+R_1}{=\!=\!=}} 3\begin{vmatrix} 9 & 2 & 3 & 1 \\ 9 & 2 & 4 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 1 & & 0 \end{vmatrix} \underset{2R_4-R_2}{=\!=} 3*\frac{1}{2}\begin{vmatrix} 1 & 2 & 3 & 1 \\ 9 & 2 & 3 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -4 & -1 \end{vmatrix}$$

$$\underset{R_4+\frac{4}{3}R_3}{=\!=\!=} \frac{3}{2}\begin{vmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -7/3 \end{vmatrix} = \frac{3}{\cancel{2}} * 1 * \cancel{2} * \cancel{3} * \left(-\frac{7}{\cancel{3}}\right) = -21 \quad \therefore \Delta = -21$$

$\boxed{\frac{7}{68}}$

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & \\ 0 & 1 & 2 & 3 \end{vmatrix} \underset{R_1-2R_2}{\overset{R_{1\,2}}{=}} - \begin{vmatrix} 8 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & -3 & -6 \\ 1 & 2 & 3 \end{vmatrix} \underset{R_4-R_2}{=} - \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -6 \\ 0 & 0 & 1 & 4 \end{vmatrix} \underset{\substack{R_3/(-3) \\ R_4+\frac{R_3}{3}}}{=} -(-3)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 6$$

$\boxed{\frac{9}{68}}$

$$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \underset{R_4-2R_3}{\overset{R_2+2R_1}{=\!=\!=}} \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \underset{R_5-R_4}{=\!=\!=} \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = -2$$

$\boxed{\frac{3}{74}}$

$$\Delta_A = (+2)*(8-3) = 10 \quad , \quad \Delta_B = +5*(-2-3) + 1*(1+7) = -25+8 = -17$$

$$AB = \begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix} \quad \therefore \Delta_{AB} = +10*(-17-3) + 2*(9+31) = -250+80 = -170$$

$$\therefore \Delta_{AB} = -170 = 10*(-17) = \Delta_A * \Delta_B.$$

$\boxed{\frac{49}{74}}$ A is invertable iff $\det A \neq 0$

but, $\det A = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 6 & 7 \\ 0 & 8 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 6 & 7 \\ 8 & -1 \end{vmatrix} = -6-56 = -62 \neq 0$

$\therefore A$ is invertable.

$\boxed{\frac{6}{75}}$ $\Delta = +(-5)*(x^2-2x) = -5x(x-2) = 0 \quad \therefore x=0$ or $x=2$

$\boxed{\frac{8}{75}}$ (a) $\Delta = (k-3)(k-2) - 4 = 0 \quad \therefore k^2-5k+6-4=0 \quad \therefore k^2-5k+2=0$

$\therefore k = \frac{5\pm\sqrt{25-8}}{2} = \frac{5\pm\sqrt{17}}{2} = 4.562$ or 0.438

(b) $\Delta = 2+12k+36-4k-12-18 = 8k+8 = 8(k+1) = 0 \quad \therefore k=-1$

$\boxed{\frac{6}{84}}$ $A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -8 \\ -1 & -3 & 4 \end{bmatrix}$

$$\therefore \quad C_A = \begin{bmatrix} +\begin{vmatrix} 0 & -8 \\ -3 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -8 \\ -1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ -1 & -3 \end{vmatrix} \\ -\begin{vmatrix} 3 & 7 \\ -3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 7 \\ -1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -1 & -3 \end{vmatrix} \\ +\begin{vmatrix} 3 & 7 \\ 0 & -8 \end{vmatrix} & -\begin{vmatrix} 1 & 7 \\ 2 & -8 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -24 & 0 & -6 \\ -33 & 11 & 0 \\ -24 & 22 & -6 \end{bmatrix}$$

$$\therefore \quad \det A = 1 * (-24) + 0 + 7(-6) = -24 - 42 = -66$$

$$\& \quad A^{-1} = \frac{C_A^T}{\det A} = \frac{1}{-66} \begin{bmatrix} -24 & -33 & -24 \\ 0 & 11 & 22 \\ -6 & 0 & -6 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{1}{2} & \frac{4}{11} \\ 0 & -\frac{1}{6} & -\frac{1}{3} \\ \frac{1}{11} & 0 & \frac{1}{11} \end{bmatrix}.$$

\# Hence: if $a = \langle 1, 2, -1 \rangle$

, $b = \langle 3, 0, -3 \rangle$

, $c = \langle 7, -8, 4 \rangle$

& $k_1 a + k_2 b + k_3 c = \langle 16, -2, -5 \rangle$;

then, $k_1 \langle 1, 2, -1 \rangle + k_2 \langle 3, 0, -3 \rangle + k_3 \langle 7, -8, 4 \rangle = \langle 16, -2, -5 \rangle$

$$\therefore \left.\begin{array}{l} k_1 + 3k_2 + 7k_3 = 16 \\ , \; 2k_1 + 0k_2 - 8k_3 = -2 \\ \& \; -k_1 - 3k_2 + 4k_3 = -5 \end{array}\right\} \Rightarrow \underbrace{\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -8 \\ -1 & -3 & 4 \end{bmatrix}}_{A \text{ as above}} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \\ -5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 16 \\ -2 \\ -5 \end{bmatrix} = -\frac{1}{66} \begin{bmatrix} -24 & -33 & -24 \\ 0 & 11 & 22 \\ -6 & 0 & -6 \end{bmatrix} \begin{bmatrix} 16 \\ -2 \\ -5 \end{bmatrix} = -\frac{1}{66} \begin{bmatrix} -198 \\ -132 \\ -66 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore \quad k_1 = 3, \quad k_2 = 2 \quad \& \quad k_3 = 1$

$\boxed{\frac{9}{85}}$ 3rd Column expansion $\therefore D = (-3) * \begin{vmatrix} 4 & 4 & 4 \\ 1 & 1 & -1 \\ 6 & 14 & 6 \end{vmatrix} + 3 *(-) \begin{vmatrix} 4 & 4 & 4 \\ 1 & 1 & -1 \\ 3 & 0 & 1 \end{vmatrix} =$

$= -3 * (\cancel{24} - \cancel{24} + 56 - 24 - 24 + 56) - 3 * (\cancel{4} - 12 - 12 - \cancel{4}) = -3 * 64 + 3 * 24$

$= -3 * 40 = -120$. $\therefore \Delta = -120$

OR: $\det A = \begin{vmatrix} 4 & 4 & 0 & 4 \\ 1 & 1 & 0 & -1 \\ 3 & 0 & -3 & 1 \\ 6 & 14 & 3 & 6 \end{vmatrix} \begin{matrix} \\ 4R_2 - R_1 \\ R_3 - 3R_2 \\ R_4 - 6R_2 \end{matrix} = \frac{1}{4} \cdot \begin{vmatrix} 4 & 4 & 0 & 4 \\ 0 & 0 & 0 & -8 \\ 0 & -3 & -3 & 4 \\ 0 & 8 & 3 & 12 \end{vmatrix} \begin{matrix} = \\ R_4 \\ 8R_3 + 3R_4 \\ R_2 \end{matrix}$

$$= \frac{1}{4} \cdot \frac{1}{8} \cdot (-1) \begin{vmatrix} 4 & 4 & 0 & 4 \\ 0 & 8 & 3 & 12 \\ 0 & 0 & -15 & 68 \\ 0 & 0 & 0 & -8 \end{vmatrix} = \frac{1}{4} \cdot \frac{1}{8} \cdot (-1) * 4 * 8 * (-15)(-8)$$

$$= -15 * 8 = -120$$

$\therefore \det A = -120$

$\boxed{\frac{12}{85}}$ $\left.\begin{matrix} 3x_1 - 4x_2 = -5 \\ 2x_1 + x_2 = 4 \end{matrix}\right\} \Longrightarrow \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$

$\therefore \Delta = \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3 + 8 = 11$

& $\Delta_{x_1} = \begin{vmatrix} -5 & -4 \\ 4 & 1 \end{vmatrix} = -5 + 16 = 11$

& $\Delta_{x_2} = \begin{vmatrix} 3 & -5 \\ 2 & 4 \end{vmatrix} = 12 + 10 = 22$

$\therefore x_1 = \frac{\Delta x_1}{\Delta} = \frac{11}{11} = 1$

& $x_2 = \frac{\Delta x_2}{\Delta} = \frac{22}{11} = 2$

$\therefore x_1 = 1$ & $x_2 = 2$

$\boxed{\dfrac{14}{85}}$ $\begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 1 & -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$ $\therefore \Delta = 4+1+8\cancel{-2}+8\cancel{+2} = 21$, $\Delta_x = \cancel{-4}+8\cancel{+4}+2+8+8 = 26$

$-\Delta_y = -4-4\cancel{+8}-1-16\cancel{+8} = -25$, $\Delta_z = \cancel{4}+2\cancel{-4}+1+8+4 = 15$

$\therefore x = \dfrac{\Delta x}{\Delta} = \dfrac{26}{21}$, $y = \dfrac{\Delta y}{\Delta} = +\dfrac{25}{21}$, $z = \Delta_z/\Delta = \dfrac{15}{21} = \dfrac{5}{7}$

$\therefore$ x, y, z are respectively $26/21$, $25/21$ & $5/7$

$\boxed{\dfrac{15}{85}}$ $\begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & -1 & 0 & -2 \\ 4 & 0 & -3 & 0 \end{bmatrix}$ is the augmented matrix $\therefore \Delta = 3+0+0+4-18-0 = -11$

$\Delta_{x_1} = 0+0+12-0-0+18 = 30$

$\Delta_{x_2} = -[0-8-24-0-0-6] = 38$, $\Delta_{x_3} = 0+24+0+16+0+0 = 40$

$\therefore x_1 = \dfrac{30}{-11}$, $x_2 = \dfrac{38}{-11}$, $x_3 = \dfrac{40}{-11}$

$\boxed{\dfrac{16}{85}}$ $\begin{bmatrix} 2 & -1 & 1 & -4 & -32 \\ 7 & 2 & 9 & -1 & 14 \\ 3 & -1 & 1 & 1 & 11 \\ 1 & 1 & -4 & -2 & -4 \end{bmatrix} \xrightarrow{\substack{R_1-2R_4 \\ R_2-7R_4 \\ R_3-3R_4}} \begin{bmatrix} 0 & -3 & 9 & 0 & -24 \\ 0 & -5 & 37 & 13 & 42 \\ 0 & -4 & 13 & 7 & 23 \\ 1 & 1 & -4 & -2 & -4 \end{bmatrix} \xrightarrow{\substack{R_3+4R_1/(-3) \\ R_2+\frac{5R_1}{-3} \\ R_1/(-3)}} \begin{bmatrix} 0 & 0 & 1 & 7 & 55 \\ 0 & 0 & 22 & 13 & 82 \\ 0 & 1 & -3 & 0 & 8 \\ 1 & 1 & -4 & -2 & -4 \end{bmatrix}$

$\xrightarrow{\substack{R_2-22R_1 \\ R_1}} \begin{bmatrix} 0 & 0 & 0 & -141 & -1128 \\ 0 & 0 & 1 & 7 & 55 \\ 0 & 1 & -3 & 0 & 8 \\ 1 & 1 & -4 & -2 & -4 \end{bmatrix}$

$\therefore -141\, x_4 = -11.28 \Rightarrow x_4 = 8$

$, x_3 + 7(8) = 55 \Rightarrow x_3 = 55-56 = -1$

$, x_2 - 3(-1) = 8 \Rightarrow x_2 = 8-3 = 5$

& $x_1 + 5 - 4(-1) - 2(8) = -4$ $\therefore x_1 = -4+16-4-5 = 3$

$\therefore$ The solution is $x_1 = 3$, $x_2 = 5$, $x_3 = -1$ & $x_4 = 8$

$\boxed{\frac{18}{85}}$ Performing determinent's reduction on the augmented matrix

$$\therefore \Delta\Delta = \begin{vmatrix} 4 & 1 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 & 1 \\ 7 & 3 & -5 & 8 & -3 \\ 1 & 1 & 1 & 2 & 3 \end{vmatrix} \begin{matrix} R_4 \\ 3R_4 - R_2 \\ R_3 - 7R_4 \\ R_1 - 4R_4 \end{matrix} = (-)(-) \begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & -4 & 4 & 5 & 8 \\ 0 & -4 & -12 & -6 & -24 \\ 0 & -3 & -3 & -7 & -6 \end{vmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ 4R_4 - 3R_2 \end{matrix} =$$

$$= \frac{1}{4} \begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & -4 & 4 & 5 & 8 \\ 0 & 0 & -16 & -11 & -32 \\ 0 & 0 & -24 & -43 & -48 \end{vmatrix} \qquad \therefore z = \frac{\Delta z}{\Delta} = \frac{\frac{1}{4}(1 * -4)\begin{vmatrix} -32 & -11 \\ -48 & -43 \end{vmatrix}}{\frac{1}{4}(1 * -4)\begin{vmatrix} -16 & -11 \\ -24 & -43 \end{vmatrix}} =$$

$$= \frac{32 * 43 - 11 * 48}{16 * 43 - 11 * 24} = 2 \qquad \therefore z = 2$$

$\boxed{\frac{24}{86}}$ The equation is $\frac{y - b_1}{x - a_1} = \frac{b_2 - b_1}{a_2 - a_1}$ or $(a_2 - a_1)y - (b_2 - b_1)x - b_1a_2 + \cancel{b_1a_1} + a_1b_2 - \cancel{a_1b_1} = 0$

or $(a_2 - a_1)y + (b_1 - b_2)x + a_1b_2 - a_2b_1 = 0$ ...... (1)

but since $\Delta = 0$ $\therefore b_1x + a_2y + a_1b_2 - a_2b_1 - ya_1 - xb_2 = 0$

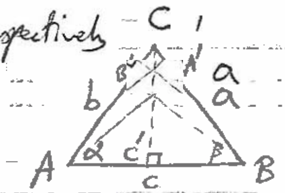
or $(a_2 - a_1)y + (b_1 - b_2)x + a_1b_2 - a_2b_1 = 0$ ...... (2)

$\therefore$ (1) = (2) $\therefore$ It could be written as such.

$\boxed{\frac{2}{86}}$ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \Delta = \cos^2\theta + \sin^2\theta = 1$, $\Delta_{x'} = x\cos\theta + y\sin\theta$, $\Delta y' = y\cos\theta - x\sin\theta$

$\therefore x' = \frac{\Delta x'}{\Delta} = x\cos\theta + y\sin\theta$, $y' = \frac{\Delta y'}{\Delta} = y\cos\theta - x\sin\theta$

$\boxed{\frac{5}{87}}$ ⓐ Project normals from A, B, C to CB, AC, BA respectively



$\therefore\ a = BA' + A'C = c\cos\beta + b\cos\gamma$

& $b = AB' + B'C = c\cos\alpha + a\cos\gamma$

& $c = AC' + C'B = b\cos\alpha + a\cos\beta$

$$\therefore \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \therefore\ \Delta = cab + bca = 2abc$$

$$, \quad \Delta_{\cos\alpha} = c^2a + b^2a - a^3 = a(b^2 + c^2 - a^2)$$

$$\therefore\ \cos\alpha = \frac{\Delta_{\cos\alpha}}{\Delta} = \frac{b^2+c^2-a^2}{2bc}$$

ⓑ $\Delta_{\cos\beta} = -b^3 + a^2b + bc^2 \qquad \therefore\ \cos\beta = \frac{\Delta_{\cos\beta}}{\Delta} = \frac{a^2+c^2-b^2}{2ac}$

& $\Delta_{\cos\gamma} = cb^2 + a^2c - c^3 \qquad \therefore\ \cos\gamma = \frac{\Delta_{\cos\gamma}}{\Delta} = \frac{a^2+b^2-c^2}{2ab}$

$\boxed{\frac{6}{87}}$

$$\left.\begin{aligned} x - 2y &= \lambda x \quad \therefore\ (1-\lambda)x - 2y = 0 \\ x - y &= \lambda y \quad \therefore\ x - (1+\lambda)y = 0 \end{aligned}\right\} \therefore \begin{bmatrix} 1-\lambda & -2 \\ 1 & -(1+\lambda) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution will only be trivial iff $\Delta \neq 0$ but $\Delta = (1-\lambda)(-(1+\lambda)) + 2$ $= -(1-\lambda^2) + 2 = 1 + \lambda^2$ always > 0 $\therefore\ \Delta \neq 0$ $\therefore$ only the trivial solution exists

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$\boxed{\frac{10}{98}}$

$$\begin{bmatrix} 2 & 1 & 3 \\ 7 & -1 & 6 \\ 8 & 3 & 11 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore\ \Delta = -22 + 48 + 63 + 24 - 77 - 36 = 0$$

$\therefore$ There is other than trivial solution

$$\begin{bmatrix} 2 & 1 & 3 \\ 7 & -1 & 6 \\ 8 & 3 & 11 \end{bmatrix} \xrightarrow[R_3 - 4R_1]{7R_1 - 2R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 9 & 9 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow[\frac{R_2}{9} + R_3]{R_2/9} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ c_2 + c_3 = 0 \qquad$ Let $c_3 = c \qquad \therefore\ c_2 = -c \quad \therefore\ 2c_1 + c_2 + 3c_3 = 0$

$\therefore\ c_1 = (-c_2 - 3c_3)/2 = (c - 3c)/2 = -c$

$\therefore\ c_1, c_2, c_3$ are $-c, -c, c$ where c is any constant.

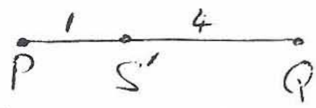
$\frac{11}{98}$ (a) Mid-point is $\frac{(2,3,-2)+(7,-4,1)}{2} = \left(\frac{9}{2}, \frac{-1}{2}, \frac{-1}{2}\right)$

(b) Let the point be S

$\therefore \frac{S-P}{Q-P} = \frac{3}{4} \quad \therefore S = \frac{3Q + P}{4} = \frac{3(7,-4,1) + (2,3,-2)}{4}$

$= \frac{(21,-12,3)+(2,3,-2)}{4} = \left(\frac{23}{4}, \frac{-9}{4}, \frac{1}{4}\right)$

\# To find s' dividing PQ by 1:4



$\langle PS' \rangle = \frac{1}{5} \langle PQ \rangle$ where $S' = (s'_1, s'_2, s'_3)$

$\therefore \langle PS' \rangle = \frac{1}{5} \langle 7-2, -4-3, 1+2 \rangle = \frac{1}{5} \langle 5, -7, 3 \rangle$

$\therefore \langle s'_1 - 2, s'_2 - 3, s'_3 + 2 \rangle = \frac{1}{5} \langle 5, -7, 3 \rangle$

$\therefore s'_1 - 2 = \frac{5}{5} = 1 \quad \therefore S'_1 = 3$

& $s'_2 - 3 = \frac{-7}{5} \quad \therefore S'_2 = 3 - \frac{7}{5} = \frac{15-7}{5} = \frac{8}{5}$

& $s'_3 + 2 = \frac{3}{5} \quad \therefore S'_3 = -2 + \frac{3}{5} = \frac{-10+3}{5} = -7/5$

$\therefore S'$ is $\left(3, \frac{8}{5}, -7/5\right)$

$\frac{3}{101}$ (a) $|\langle 1,-3,2\rangle + \langle 1,1,0\rangle| = |\langle 2,-2,2\rangle| = 2\sqrt{3} = 3.46$

(b) $|u| + |v| = \sqrt{1+9+4} + \sqrt{1+1} = \sqrt{14} + \sqrt{2} = 5.16$

(c) $|-2u| + 2|u| = 2|u| + 2|u| = 4|u| = 4\sqrt{14} = 14.97$

(d) $|3u + w - 5v| = |\langle 2,2,-4\rangle - 5\langle 1,1,0\rangle + 3\langle 1,-3,2\rangle| = |\langle 0,-12,2\rangle|$

$= \sqrt{12^2+2^2} = \sqrt{144+4} = \sqrt{148} = 12.17$

(e) $\frac{w}{|w|} = \frac{\langle 2,2,-4\rangle}{\sqrt{4+4+16}} = \frac{\langle 2,2,-4\rangle}{\sqrt{24}} = \langle .408, .408, -.816\rangle$

(f) $\left|\frac{w}{|w|}\right| = 1$

$\frac{6}{101}$ Let $v = \langle v_1, v_2, v_3\rangle \quad \therefore |v| = \sqrt{v_1^2+v_2^2+v_3^2} \quad \therefore \frac{v}{|v|} = \left\langle \frac{v_1, v_2, v_3}{\sqrt{v_1^2+v_2^2+v_3^2}} \right\rangle$

$\therefore \left|\frac{v}{|v|}\right| = \sqrt{\left(\frac{v_1}{\sqrt{v_1^2+v_2^2+v_3^2}}\right)^2 + \left(\frac{v_2}{\sqrt{v_1^2+v_2^2+v_3^2}}\right)^2 + \left(\frac{v_3}{\sqrt{v_1^2+v_2^2+v_3^2}}\right)^2} = \sqrt{\frac{v_1^2+v_2^2+v_3^2}{v_1^2+v_2^2+v_3^2}} = 1.$

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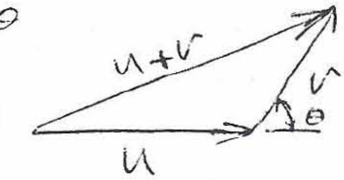
$\boxed{\frac{9}{101}}$ $|u+v|^2 = |u|^2 + |v|^2 + 2|u|\cdot|v|\cdot\cos\theta$

$$= (|u|+|v|)^2 - 2|u|\cdot|v|(1-\cos\theta)$$



$\because \cos\theta \le 1 \implies 1-\cos\theta \geq 0$

$\therefore 2|u|\cdot|v|(1-\cos\theta) \geq 0 \quad \therefore -2|u|\cdot|v|\cdot(1-\cos\theta) \le 0$

$\therefore |u+v|^2 = (|u|+|v|)^2 - 2|u|\cdot|v|\cdot(1-\cos\theta) \le (|u|+|v|)^2$

$\therefore |u+v|^2 \le (|u|+|v|)^2 \quad \therefore |u+v| \le |u|+|v|$

$\therefore$ OK

OR: Another proof using dot-product

$$|u+v|^2 = (u+v)\cdot(u+v) = u\cdot u + u\cdot v + v\cdot u + v\cdot v =$$
$$= |u|^2 + 2u\cdot v + |v|^2 = |u|^2 + |v|^2 + 2|u|\cdot|v|\cdot\cos\angle\theta_u^v$$

$$\le (\text{as in above})\ |u|^2 + |v|^2 + 2|u|\cdot|v| = (|u|+|v|)^2$$

$\therefore |u+v| \le |u|+|v| \quad \therefore$ OK

$\boxed{\frac{4c}{110}}$ $Proj_a u = (u\cdot a)\left(\frac{1}{|a|}\right)\cdot\hat{a}$ where $\hat{a} = \frac{a}{|a|}$

$$= \frac{u\cdot a}{|a|^2}\cdot a$$

$$= \frac{\langle -7,1,3\rangle\cdot\langle 5,0,1\rangle}{(5^2+0^2+1^2)}\cdot\langle 5,0,1\rangle$$

$$= \frac{-35+0+3}{25+0+1}\cdot\langle 5,0,1\rangle = \frac{-32}{26}\cdot\langle 5,0,1\rangle$$

$\therefore |Proj_a u| = \left|\frac{-32}{26}\cdot\langle 5,0,1\rangle\right| = \frac{32}{26}\cdot\sqrt{25+1} = \frac{32}{\sqrt{26}}$

$\boxed{\frac{9d}{110}}$ $\left(|\langle 1,2\rangle|\,\langle 4,-2\rangle\right)\cdot\langle 6,0\rangle = \sqrt{5}\,\langle 4,-2\rangle\cdot\langle 6,0\rangle$

$= \sqrt{5} * 24 = 24\sqrt{5}$

$\boxed{\frac{11}{110}}$ Let A be $(-1,0)$, $B(-2,1)$, $C(1,4)$

$\therefore$ $AB = \langle -1,1\rangle$, $BC = \langle 3,3\rangle$, $CA = \langle -2,-4\rangle$

$\therefore$ $|AB| = \sqrt{2}$, $|BC| = 3\sqrt{2}$, $|CA| = 2\sqrt{5}$

$\therefore$ $\langle AB\rangle\cdot\langle BC\rangle = -3+3 = 0 = \sqrt{2} * 3\sqrt{2} * \cos\angle_{AB}^{BC}$ $\therefore AB \perp BC$

& $\langle BC\rangle\cdot\langle CA\rangle = -6-12 = -18 = 3\sqrt{2} * 2\sqrt{5} * \cos\angle_{BC}^{CA}$

$\therefore$ $\angle_{BC}^{CA} = \cos^{-1}\left(\frac{-18}{6\sqrt{10}}\right) = 161.6^\circ$ $\therefore$ $\hat{BCA} = 180 - \angle_{BC}^{CA} = 18.4^\circ$

& $\langle CA\rangle\cdot\langle AB\rangle = 2-4 = -2 = \sqrt{2} * 2\sqrt{5}\cos\angle_{CA}^{AB}$

$\therefore$ $\angle_{CA}^{AB} = \cos^{-1}\left(\frac{-2}{2\sqrt{10}}\right) = 108.4^\circ$ $\therefore$ $\hat{CAB} = 180 - \angle_{CA}^{AB} = 71.6^\circ$

$\therefore$ $\hat{ABC} = 90^\circ$, $\hat{BCA} = 18.4^\circ$ & $\hat{CAB} = 71.6^\circ$.

$\boxed{\frac{12}{110}}$ $\langle AB\rangle = \langle 1,3,-2\rangle$, $\langle BC\rangle = \langle 4,-2,-1\rangle$ & $\langle CA\rangle = \langle -5,-1,3\rangle$

$\langle AB\rangle\cdot\langle BC\rangle = 4-6+2 = 0$ $\therefore$ $\hat{ABC} = 90^\circ$ $\therefore$ ΔABC is right at B.

$\boxed{\frac{15b}{110}}$ Distance $= \frac{|5+2(-3)-1|}{\sqrt{1^2+2^2}} = \frac{2}{\sqrt{5}}$

$\boxed{\frac{16}{111}}$ since $|a|^2 = a\cdot a$

$\therefore$ $|u+v|^2 = (u+v)\cdot(u+v) = u\cdot u + v\cdot u + u\cdot v + v\cdot v = |u|^2+|v|^2+2u\cdot v$

& $|u-v|^2 = (u-v)\cdot(u-v) = u\cdot u - v\cdot u - u\cdot v + v\cdot v = |u|^2+|v|^2-2u\cdot v$

$\therefore$ $|u+v|^2 + |u-v|^2 = 2|u|^2 + 2|v|^2$

$\boxed{\frac{1f}{118}}$ $\left(\langle 2,-1,3\rangle \times \langle 0,1,7\rangle\right) - 2\langle 1,4,5\rangle =$

$= \langle -10,-14,2\rangle - \langle 2,8,10\rangle = \langle -12,-22,-8\rangle$

$\boxed{\frac{3b}{118}}$ $PQ = \langle -1, 4, 8\rangle$, $QR = \langle 6, -3, 4\rangle$ $\therefore$ area $= \frac{1}{2}|\langle PQ\rangle \times \langle QR\rangle| =$

$= \frac{1}{2}|\langle 16+16, 48+4, 2-24\rangle| = \frac{1}{2}\sqrt{32^2+52^2+22^2} = \frac{\sqrt{4212}}{2} = 32.45$

$\boxed{\frac{8}{119}}$ $u\cdot(v \times w) = \langle u_1, u_2, u_3\rangle \cdot \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1\rangle =$

$= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) =$

$= u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1 - u_2v_1w_3 - u_1v_3w_2 = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$\boxed{\frac{9}{119}}$ Volume of parallelopepid = (side vector 1 $\times$ side vector 2) $\cdot$ Side vector 3

$= \langle -1, 4, 7\rangle \times \langle 6, -7, 3\rangle \cdot \langle 4, 0, 1\rangle$

$= \langle 12+49, 42+3, 7-24\rangle \cdot \langle 4, 0, 1\rangle$

$= \langle 61, 45, -17\rangle \cdot \langle 4, 0, 1\rangle = 61*4 + 0 - 17 = 227$

$\boxed{\frac{11}{119}}$ (a) $(u + kv)\times v = u\times v + kv\times v = u\times v + k*|v|*|v|*\sin 0 = u\times v$

(b) $(u\times v)\cdot z =$ (as in $\boxed{\frac{8}{119}}$) $\begin{vmatrix} \langle z\rangle \\ \langle u\rangle \\ \langle v\rangle \end{vmatrix} = -\begin{vmatrix} \langle u\rangle \\ \langle z\rangle \\ \langle v\rangle \end{vmatrix} = (-)(-)\begin{vmatrix} \langle u\rangle \\ \langle v\rangle \\ \langle z\rangle \end{vmatrix} =$

$= u\cdot(v\times z)$

$\boxed{\frac{46}{128}}$ (a) $ax + by + cz = 1$ $\therefore \begin{bmatrix} -2 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\therefore \Delta = 5, \Delta_a = 0, \Delta_b = +10, \Delta_c = -$

$\therefore a = 0, b = 2, c = -1$

$\therefore$ The equation is $2y - z = 1$

(b) $ax + by + cz = 1$ $\therefore \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\therefore \Delta = 16, \Delta_a = 1, \Delta_b = +9, \Delta_c = -5$

$\therefore a = \frac{1}{16}, b = \frac{+9}{16}$ & $c = \frac{-5}{16}$

$\therefore$ The equation is $x + 9y - 5z = 16$

$\boxed{\frac{7}{128}}$ (a) $\langle 1, -1, 3\rangle \cdot \langle 2, 0, 1\rangle = 2 + 3 = 5$ $\therefore$ No

(b) $\langle 3, -2, 1\rangle \cdot \langle 4, 5, -2\rangle = 12 - 10 - 2 = 0$ $\therefore$ Yes

$\boxed{\frac{19}{129}}$ $\langle 5, -2, 1\rangle$ is the normal

$\therefore$ $5(x-2)-2(y+7)+(z-6)=0$

$\therefore$ The eqn. of the plane is $5x-2y+z-30=0$

distance between both planes $= \left|\dfrac{5(2)-2(-7)+(6)-9}{\sqrt{5^2+2^2+1^2}}\right| =$

$= \dfrac{21}{\sqrt{30}} = \dfrac{21\sqrt{30}}{30} = \dfrac{7}{10}\sqrt{30}$

$\boxed{\frac{20}{129}}$ $x=4+5t$, $y=-2+t$, $z=4-t$

$\therefore$ $3x-y+7z+8=0$ $\therefore$ $3(4+5t)-(-2+t)+7(4-t)+8=0$

$\therefore$ $12+15t+2-t+28-7t+8=0 \Rightarrow 50+7t=0$ $\therefore t=\frac{-50}{7}$

$\therefore$ $x=4+5t=4-\frac{250}{7}=\frac{-222}{7}$, $y=-2+\frac{-50}{7}=\frac{-64}{7}$, $z=4+\frac{50}{7}=\frac{78}{7}$

$\therefore$ The point of intersection is $\left(\frac{-222}{7}, \frac{-64}{7}, \frac{78}{7}\right)$

$\boxed{\frac{21}{129}}$ put $t=0$ $\therefore$ $(-2,4,3)$ is on the plane

$\langle 3,2,-1\rangle$ is on the plane & so is $\langle 1,-2,1\rangle$

$\therefore$ normal is $\langle 0,-4,-8\rangle$

$\therefore$ The equation is $0(x+2)-4(y-4)-8(z-3)=0$ or $y-4+2(z-3)=$

or $y+2z=10$

$\boxed{\frac{22}{129}}$ Solve the two planes for line of intersection

$$\therefore \begin{bmatrix} 4 & -1 & 1 & -2 \\ 2 & 1 & -2 & -3 \end{bmatrix} \xRightarrow{R_1 - 2R_2} \begin{bmatrix} 4 & -1 & 1 & -2 \\ 0 & -3 & 5 & 4 \end{bmatrix}$$

$\therefore \; -3y + 5z + 4 = 0$ let $z = t$

$\therefore \; y = \frac{4+5t}{3}$

$\therefore \; 4x - y + z - 2 = 0 \quad \therefore x = \frac{2 - t + \frac{4+5t}{3}}{4} = \frac{6 - 3t + 4 + 5t}{12} =$

$= \frac{10 + 2t}{12} = \frac{5+t}{6}$

$\therefore$ Line of intersection is $x = \frac{5+t}{6}, \; y = \frac{4+5t}{3}, \; z = t$

direction is $\langle \frac{1}{6}, \frac{5}{3}, 1 \rangle$ or $\langle 1, 10, 6 \rangle$ in the plane

point is at $t = 1$ $\quad (1, 3, 1)$

another direction in the plane is $\langle (1,3,1) - (-1,4,2) \rangle = \langle 2, -1, -1 \rangle$

$\therefore$ normal is $\langle 2, -1, -1 \rangle \times \langle 1, 10, 6 \rangle = \langle 4, -13, 21 \rangle$

$\therefore$ The plane is $4(x-1) - 13(y-3) + 21(z-1) = 0$

OR $\quad 4x - 13y + 21z + 14 = 0$

$\boxed{\frac{23}{129}}$ $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ & $D(4,2,3)$

Normal to $ABC = \langle AB\rangle \times \langle BC\rangle = \langle -1,2,4\rangle \times \langle -2,-1,-2\rangle$

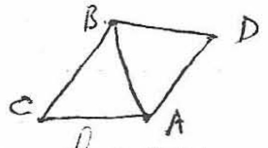
$= \langle -4+4, -8-2, 1+4\rangle = \langle 0,-10,5\rangle = 5\langle 0,-2,1\rangle$

Normal to $ABD = \langle AB\rangle \times \langle BD\rangle = \langle -1,2,4\rangle \times \langle 4,0,0\rangle$

$= \langle 0,16,-8\rangle = -8\langle 0,-2,1\rangle$

$\therefore$ ABC plane // ABD plane with points in common.

$\therefore$ ABCD lie in one plane. Order is shown



$\#$ Area of ACBD = area of ΔABC + area of ΔABD

$= \frac{1}{2}\left|\langle AB\rangle \times \langle BC\rangle\right| + \frac{1}{2}\left|\langle AB\rangle \times \langle BD\rangle\right| = \frac{1}{2}\left|5\langle 0,-2,1\rangle\right| +$

$\frac{1}{2}\left|-8\langle 0,-2,1\rangle\right| = \left(\frac{5}{2}+\frac{8}{2}\right)\sqrt{0^2+2^2+1^2} = \frac{13}{2}\cdot\sqrt{5} = \frac{13\sqrt{5}}{2}$

$\#$ Equation of plane is $ax+by+cz=1$

$$\left.\begin{array}{lcl} (1,0,-1) & \Rightarrow & a \quad\quad -c=1 \\ (0,2,3) & \Rightarrow & 2b+3c=1 \\ (-2,1,1) & \Rightarrow & -2a+b+c=1 \\ (4,2,3) & \Rightarrow & 4a+2b+3c=1 \end{array}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ -2 & 1 & 1 & 1 \\ 4 & 2 & 3 & 1 \end{bmatrix}$$

If system is consistant, then, they are on the same plane

$$\xrightarrow[\substack{2R_1+R_3 \\ 2R_3+R_4}]{} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 4 & 5 & 3 \end{bmatrix} \xrightarrow[\substack{2R_3-R_2 \\ 2R_2-R_4}]{} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -5 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[\substack{R_3/5 \\ 5R_4+R_3}]{} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ System Consistant $\therefore$ on same plane

$-c=1 \quad \therefore c=-1, \quad b=(1-3c)/2=(1+3)/2=4/2=2$

$a=1+c=1-1=0$

$\therefore$ Equation of plane is $2y-z=1$

$\boxed{\frac{24}{129}}$ $n_1 = \langle 1, -4, 2\rangle$, $n_2 = \langle 2, 3, -1\rangle$ ∴ $n_1 \times n_2 = \ell = \langle -2, 5, 11\rangle$

∴ equation of line is $\frac{x-5}{-2} = \frac{y-0}{5} = \frac{z+2}{11} = t$

∴ The parametric equations are:

$x = 5 - 2t$

$y = 5t$

$z = -2 + 11t$

$\boxed{\frac{26}{130}}$ normal to plane // line of intersection $= \langle 2, 1, 1\rangle \times \langle 1, 2, 1\rangle$

$= \langle -1, -1, 3\rangle$

∴ $-1(x-1) - 1(y-2) + 3(z+1) = 0$ is the eqn. of plane

∴ $-x - y + 3z + 1 + 2 + 3 = 0$

∴ The equation of the plane is $x + y - 3z = 6$

$\boxed{\frac{27}{130}}$ $P_1P_2 = \langle 3, -1, -1\rangle$, $n' = \langle 4, -1, 3\rangle$ ∴ $n = \langle -4, -13, 1\rangle$

∴ $-4(x+2) - 13(y-1) + z - 4 = 0$

∴ The equation is $4x + 13y - z = 1$

$\boxed{\frac{28}{130}}$ $\ell_1 : \langle 1, 2, -1\rangle$, $\ell_2 : \langle -1, -2, 1\rangle = -\langle 1, 2, -1\rangle$

∴ $\ell_1 // \ell_2$

put $t = 0$ ∴ $(-2, 3, 4)$ on ℓ_1 & $(3, 4, 0)$ on ℓ_2 are on the plane. ∴ The vector joining them $= \langle 5, 1, -4\rangle$ is also on the plane ∴ The normal is $\langle 5, 1, -4\rangle \times \langle 1, 2, -1\rangle = \langle 7, 1, 9\rangle$

∴ The equation is $7(x+2) + 1(y-3) + 9(z-4) = 0$

or $7x + y + 9z = 25$.

$\boxed{\frac{29}{130}}$ Put $t=0$ $\therefore$ $(-1,0,-4)$ lie on the plane

$\therefore$ Normal to plane is $\langle(2,0,3)-(-1,0,-4)\rangle \times \langle 1,1,2\rangle$

$= \langle 3,0,7\rangle \times \langle 1,1,2\rangle = \langle -7,1,3\rangle$

$\therefore$ Eq. is $-7(x+1)+y+3(z+4)=0$

or. $-7x+y+3z+5=0$

$\boxed{\frac{31}{130}}$ line of intersection, $l = \langle 2,-1,1\rangle \times \langle 0,1,1\rangle = \langle -2,-2,2\rangle$

$\therefore n = \langle -2,-2,2\rangle \times \langle 3,1,2\rangle = \langle -6,10,4\rangle$

put $t=0$ $\therefore$ $(0,1,0)$ is on the plane

$\therefore -6(x-0)+10(y-1)+4(z-0)=0$

$\therefore -6x+10y+4z=10$

$\therefore$ The equation of the plane is $3x-5y-2z+5=0$

$\boxed{\frac{33}{130}}$ l_1 is $\frac{x+1}{4} = \frac{y-3}{1} = \frac{z-1}{0} = t$

l_2 is $\frac{x+13}{12} = \frac{y-1}{6} = \frac{z-2}{3} = t'$

z of $l_1 = 1$ $\therefore$ t' of $l_2 = \frac{1-2}{3} = \frac{-1}{3}$ $\therefore y = 1+6t' = 1-2=-$

$\therefore t$ of $l_1 = \frac{-1-3}{1} = -4$ $\therefore x = -1+4*t = -1-16 = -17$

check with x of l_2 $\therefore x = -13+12t' = -13+12*(-\frac{1}{3}) = -13-4=-1$

$\therefore$ The two lines l_1 & l_2 intersect at $(-17,-1,1)$

OR: $x+1=4t$ (1), $y-3=t$ (2), $z-1=0$ (3) for line one l_1

$x+13=12r$ (4), $y-1=6r$ (5), $z-2=3r$ (6) " " two l_2

six equations with five unknowns, if consistent then they intersect.

from (3) $\therefore z=1$ into (6) $\therefore r = \frac{1-2}{3} = -\frac{1}{3}$

into (5) $\therefore y = 1+6(\frac{1}{3}) = -1$ into (2) $\therefore t = -1-3=-4$

into (1) $\therefore x = -1+4(-4) = -17$

But from (4) $x = -13+12(-\frac{1}{3}) = -17$ $\therefore$ OK, intersect.

$\therefore$ Point of intersection is $(-17,-1,1)$

34/130

They intersect at $(-17, -1, 1)$

$l_1 = \langle 4, 1, 0\rangle$, $l_2 = \langle 12, 6, 3\rangle$ $\therefore n = l_1 \times l_2 = \langle 3, -12,$

$\therefore$ The equation is $3(x+17) - 12(y+1) + 12(z-1) = 0$

or $x + 17 - 4(y+1) + 4(z-1) = 0 \Longrightarrow x - 4y + 4z + 9 = 0$

38c/131

$P_1: x + y + z = 1$, $P_2: x + y + z = -1 \Longrightarrow P_2: x + y + z + 1 = 0$

$(0, 0, 1)$ is on P_1

$\therefore$ distance between P_1 & P_2 $= \dfrac{0+0+1+1}{\sqrt{1^2+1^2+1^2}} = \dfrac{2}{\sqrt{3}}$

8/139

$|kv| = \sqrt{k^2}\sqrt{1+4+0+9} = \sqrt{14}\sqrt{k^2} = 3 \quad \therefore k = \pm\dfrac{3}{\sqrt{14}} = \pm 0.802$

9c/139

$u \cdot v = \langle 1, -1, 2, 3\rangle \cdot \langle 3, 3, -6, 4\rangle = 3 - 3 - 12 + 12 = 0$

11c/139

$|u - v| = |\langle 2+1, 0-4, 1-6, 3-6\rangle| =$

$= |\langle 3, -4, -5, -3\rangle| = \sqrt{3^2 + 4^2 + 5^2 + 3^2}$

$= \sqrt{9 + 16 + 25 + 9} = \sqrt{59}$

& $u \cdot v = -2 + 0 + 6 + 18 = 22$

11d/139

$|u - v| = |\langle 6, 0, 1, 3, 0\rangle - \langle -1, 4, 2, 8, 3\rangle| = |\langle 7, -4, -1, -5, -$

$= \sqrt{49 + 16 + 1 + 25 + 9} = \sqrt{100} = 10$

12/139

$|u+v|^2 + |u-v|^2 = (u+v)\cdot(u+v) + (u-v)\cdot(u-v) =$

$= (u\cdot u + u\cdot v + v\cdot u + v\cdot v) + (u\cdot u - u\cdot v - v\cdot u + v\cdot v)$

$= |u|^2 + 2u\cdot v + |v|^2 + |u|^2 - 2u\cdot v + |v|^2$

$= 2|u|^2 + 2|v|^2 \quad \therefore$ OK

$\boxed{\frac{13}{139}}$

$|u \pm v|^2 = \langle u \pm v \rangle \cdot \langle u \pm v \rangle = u \cdot \langle u \pm v \rangle \pm v \cdot \langle u \pm v \rangle =$

$= u \cdot u \pm u \cdot v \pm v \cdot u + v \cdot v = |u|^2 \pm 2u \cdot v + |v|^2$

$\therefore |u+v|^2 - |u-v|^2 = |u|^2 + 2u \cdot v + |v|^2 - (|u|^2 - 2u \cdot v + |v|^2) = 4u \cdot v$

$\therefore u \cdot v = \frac{1}{4}|u+v|^2 - \frac{1}{4}|u-v|^2$

$\boxed{\frac{8}{144}}$ $\langle x,y \rangle + \langle x',y' \rangle = \langle x+x'+1, y+y'+1 \rangle$ & $k\langle x,y \rangle = \langle kx,ky \rangle$

The axioms are:

(1) $\langle x,y \rangle + \langle x',y' \rangle = \langle x+x'+1, y+y'+1 \rangle$ which is in the space, ok

(2) $\langle x,y \rangle + \langle x',y' \rangle = \langle x+x'+1, y+y'+1 \rangle$

$\langle x',y' \rangle + \langle x,y \rangle = \langle x'+x+1, y'+y+1 \rangle$ $\therefore \langle x,y \rangle + \langle x',y' \rangle = \langle x',y' \rangle + \langle x,y \rangle$

$\therefore$ OK

(3) $\langle x,y \rangle + (\langle x',y' \rangle + \langle x'',y'' \rangle) = \langle x,y \rangle + \langle x'+x''+1, y'+y''+1 \rangle$

$= \langle x+x'+x''+2, y+y'+y''+2 \rangle$

$(\langle x,y \rangle + \langle x',y' \rangle) + \langle x'',y'' \rangle = \langle x+x'+1, y+y'+1 \rangle + \langle x'',y'' \rangle$

$= \langle x+x'+x''+2, y+y'+y''+2 \rangle$ $\therefore$ OK

(4) Assuming 0 to be $\langle -1,-1 \rangle$

$\therefore \langle x,y \rangle + 0 = \langle x,y \rangle + \langle -1,-1 \rangle = \langle x-1+1, y-1+1 \rangle = \langle x,y \rangle$, ok

(5) Define the negative of $u\langle x,y \rangle$ to be $\langle -x-2, -y-2 \rangle$

$\therefore \langle x,y \rangle + \langle -x-2, -y-2 \rangle = \langle x-x-2+1, y-y-2+1 \rangle = \langle -1,-1 \rangle = 0$, ok

(6) $k\langle x,y \rangle = \langle kx, ky \rangle$ which is in the space, OK

(7) $k(\langle x,y \rangle + \langle x',y' \rangle) = k\langle x+x'+1, y+y'+1 \rangle = \langle kx+kx'+k, ky+ky'+k \rangle$

$k\langle x,y \rangle + k\langle x',y' \rangle = \langle kx,ky \rangle + \langle kx',ky' \rangle = \langle kx+kx'+1, ky+ky'+1 \rangle$

$\therefore$ This axiom fails.

(8) $(k+l)\langle x,y \rangle = \langle (k+l)x, (k+l)y \rangle = \langle kx+lx, ky+ly \rangle$

$k\langle x,y \rangle + l\langle x,y \rangle = \langle kx,ky \rangle + \langle lx,ly \rangle = \langle kx+lx+1, ky+ly+1 \rangle$

$\therefore$ This axiom fails.

(9) $k(l\langle x,y \rangle) = k\langle lx,ly \rangle = \langle klx, kly \rangle$

$kl\langle x,y \rangle = \langle klx, kly \rangle$, ok

(10) $1\langle x,y \rangle = \langle x,y \rangle$, ok

$\therefore$ The set is not a vector space because axioms (7) & (8) fail.

$\boxed{\frac{9}{144}}$ (1) $x + x' = xx' \in R^+$ $\therefore$ OK

(2) $x + x' = xx'$ & $x' + x = x'x$ $\therefore x + x' = x' + x$ $\therefore$ OK

(3) $(x + x') + x'' = (xx') + x'' = xx'x''$

& $x + (x' + x'') = x + (x'x'') = xx'x''$

$\therefore (x + x') + x'' = x + (x' + x'')$ $\therefore$ OK

(4) Define the value 1 as zero element, $\underline{0}$.

$\therefore \underline{0} + x = 1 + x = 1 \cdot x = x$

& $x + \underline{0} = x + 1 = x \cdot 1 = x$ $\therefore \underline{0} + x = x + \underline{0} = x$ $\therefore$ OK

(6) $kx = x^k \in R^+$ $\therefore$ OK

(7) $k(x + x') = k(xx') = (xx')^k = x^k \cdot x'^k$

& $kx + kx' = x^k + x'^k = x^k \cdot x'^k$

$\therefore k(x + x') = kx + kx'$ $\therefore$ OK

(5) Define the reciprocal as negative element $(x \neq 0)$

$\therefore x + (-x) = x + \frac{1}{x} = x \cdot \frac{1}{x} = 1 = \underline{0}$ $\therefore$ OK

(8) $(k + l)x = x^{k+l} = x^k \cdot x^l$

& $kx + lx = x^k + x^l = x^k \cdot x^l$

$\therefore (k + l)x = kx + lx$ $\therefore$ OK

(9) $k(lx) = k(x^l) = (x^l)^k = x^{lk}$

& $(kl)(x) = x^{kl} = x^{lk}$

$\therefore k(lx) = (kl)(x)$ $\therefore$ OK

(10) $1 \cdot u = u^1 = u$ $\therefore$ OK

$\therefore$ All axioms are satisfied, hence, the set is a vector space for all positive real numbers, R^+

$\frac{10}{144}$. Let $A \in \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ defines S

The axioms are:

(1) $A + A' = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} a' & 1 \\ 1 & b' \end{bmatrix} = \begin{bmatrix} a+a' & 2 \\ 2 & b+b' \end{bmatrix} \notin S$. This axiom fails

(2) $A + A' = \begin{bmatrix} a+a' & 2 \\ 2 & b+b' \end{bmatrix}$, $A' + A = \begin{bmatrix} a'+a & 2 \\ 2 & b'+b \end{bmatrix}$, OK

(3) $A + (A' + A'') = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \left(\begin{bmatrix} a' & 1 \\ 1 & b' \end{bmatrix} + \begin{bmatrix} a'' & 1 \\ 1 & b'' \end{bmatrix} \right) = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} a'+a'' & 2 \\ 2 & b'+b'' \end{bmatrix} =$

$= \begin{bmatrix} a+a'+a'' & 3 \\ 3 & b+b'+b'' \end{bmatrix}$, $(A + A') + A'' = \begin{bmatrix} a+a' & 2 \\ 2 & b+b' \end{bmatrix} + \begin{bmatrix} a'' & 1 \\ 1 & b'' \end{bmatrix} = \begin{bmatrix} a+a'+a'' & 3 \\ 3 & b+b'+ \end{bmatrix}$

$\therefore$ OK

(4) Let the 0 object be $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore 0 + A = A$ but $0 \notin S$ $\therefore$ Thi
axiom fails.

(5) Let $-A$ be $\begin{bmatrix} -a & -1 \\ -1 & -b \end{bmatrix}$ $\therefore A + (-A) = 0$ but then $-A \notin S$,
$\therefore$ This axiom fails.

(6) $kA = \begin{bmatrix} ka & k \\ k & kb \end{bmatrix} \notin S$ $\therefore$ This axiom fails.

(7) $k(A + A') = k\begin{bmatrix} a+a' & 2 \\ 2 & b+b' \end{bmatrix} = \begin{bmatrix} ka+ka' & 2k \\ 2k & kb+kb' \end{bmatrix}$, $kA + kA' = \begin{bmatrix} ka & k \\ k & kb \end{bmatrix} + \begin{bmatrix} ka' & \\ k & \end{bmatrix}$

$= \begin{bmatrix} ka+ka' & 2k \\ 2k & kb+kb' \end{bmatrix}$, OK.

(8) $(k+l)A = \begin{pmatrix} (k+l)a & k+l \\ k+l & (k+l)b \end{pmatrix} = \begin{pmatrix} ka+la & k+l \\ k+l & kb+lb \end{pmatrix}$, $kA + lA = \begin{pmatrix} ka & k \\ k & kb \end{pmatrix}$
$\begin{pmatrix} la & l \\ l & lb \end{pmatrix} = \begin{pmatrix} ka+la & k+l \\ k+l & kb+lb \end{pmatrix}$, OK

(9) $k(lA) = k\begin{pmatrix} la & l \\ l & lb \end{pmatrix} = \begin{pmatrix} kla & kl \\ kl & klb \end{pmatrix}$, $(kl)A = \begin{pmatrix} kla & kl \\ kl & klb \end{pmatrix}$, OK.

(10) $1A = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = A$, OK

$\therefore$ The set is not a vector space because axioms (1), (4), (5) & (6) f

$\boxed{\frac{14}{144}}$ Let u be moon $\in S$, the axioms are

(1) $u+v =$ moon + moon = moon $\in S$, OK

(2) $u+v =$ moon + moon = moon, $v+u =$ moon + moon = moon, OK

(3) $u+(v+w) =$ moon + (moon + moon) = moon + moon = moon

$(u+v)+w =$ (moon + moon) + moon = moon + moon = moon, OK

(4) Let 0 be moon

$\therefore$ $0+u =$ moon + moon = moon = u, OK

(5) Let $-u$ be moon

$\therefore$ $u+(-u) =$ moon + moon = moon = 0, OK

(6) $ku = k *$ moon = moon $\in S$, OK

(7) $k(u+v) = k$(moon + moon) = k(moon) = moon

$ku + kv = k$(moon) + k(moon) = moon + moon = moon, OK

(8) $(k+l)u = (k+l)$(moon) = moon, $ku + lu = k$(moon) + l(moon) = moon + moon = moon, OK

(9) $k(lu) = k(l$(moon)$) = k$(moon) = moon, (kl)(moon) = moon, $\therefore$ OK

(10) 1(moon) = moon, OK

$\therefore$ The set is a vector space because it satisfy all axioms.

$\boxed{\frac{6a}{152}}$ $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 1 & -1 & 2 & 9 \\ 4 & 3 & 5 & 5 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_1-2R_2} \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 3 & -1 & -13 \\ 0 & 1 & -1 & -5 \end{bmatrix} \xrightarrow{R_2-3R_3} \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 3 & -1 & -13 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

$\therefore 2c_3 = 2 \therefore c_3 = 1$ & $3c_2 - c_3 = -13 \therefore c_2 = (-13+1)/3 = -4$

& $2c_1 + c_2 + 3c_3 = 5 \therefore 2c_1 + (-4) + 3(1) = 5 \therefore c_1 = \frac{5-3+4}{2} = 3$

$\therefore \langle 5, 9, 5 \rangle = 3\langle 2, 1, 4\rangle - 4\langle 1, -1, 3\rangle + \langle 3, 2, 5\rangle$

$\therefore$ The vector is $3u - 4v + w$

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$$\begin{bmatrix} 2 & 1 & 3 & 2 \\ 1 & -1 & 2 & 2 \\ 4 & 3 & 5 & 3 \end{bmatrix} \xRightarrow[R_3-2R_1]{R_1-2R_2} \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & 3 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{bmatrix} \xRightarrow{R_2-3R_3} \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$\therefore c_3 = \frac{1}{2}$, $c_2 = (-2+\frac{1}{2})/3 = -\frac{3}{2\times 3} = -\frac{1}{2}$, $c_1 = [2-3(\frac{1}{2})-(-\frac{1}{2})]/2 = \frac{1}{2}$

$\therefore \langle 2, 2, 3\rangle = +\frac{1}{2}\langle 2, 1, 4\rangle - \frac{1}{2}\langle 1, -1, 3\rangle + \frac{1}{2}\langle 3, 2, 5\rangle$

$\therefore 2 + 2x + 3x^2 = \frac{1}{2}P_1 - \frac{1}{2}P_2 + \frac{1}{2}P_3$

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$$\begin{bmatrix} 1 & 0 & 4 & 6 \\ 2 & 1 & -2 & 3 \\ -1 & 2 & 0 & 0 \\ 3 & 4 & -2 & 8 \end{bmatrix} \xRightarrow[\substack{R_3+R_1 \\ R_4-3R_1}]{R_2-2R_1} \begin{bmatrix} 1 & 0 & 4 & 6 \\ 0 & 1 & -10 & -9 \\ 0 & 2 & -4 & -6 \\ 0 & 4 & -14 & -10 \end{bmatrix} \xRightarrow[R_4-4R_2]{R_3-2R_2} \begin{bmatrix} 1 & 0 & 4 & 6 \\ 0 & 1 & -10 & -9 \\ 0 & 0 & 24 & 24 \\ 0 & 0 & 26 & 26 \end{bmatrix} \xRightarrow[\frac{R_4}{26}-\frac{R_3}{24}]{R_3/24}$$

$$\begin{bmatrix} 1 & 0 & 4 & 6 \\ 0 & 1 & -10 & -9 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \therefore c_3 = 1, \quad c_2 = -9 + 10c_3 = 1, \quad c_1 = 6 - 4c_3 =$$

$$\therefore \begin{bmatrix} 6 & 3 \\ 0 & 8 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 0 & -2 \end{bmatrix}$$

$\therefore \begin{bmatrix} 6 & 3 \\ 0 & 8 \end{bmatrix} = 2A + B + C$ (it is Linear combination of A, B & C)

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(a) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -6 \neq 0 \quad \therefore V_1, V_2, V_3$ span R^3

(b) $\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 16 - 12 - 16 - 24 + 32 + 4 = 0 \quad \therefore V_1, V_2, V_3$ do not span R

(c) $$\begin{bmatrix} 3 & 2 & 5 & 1 \\ 1 & -3 & -2 & -4 \\ 4 & 5 & 9 & -1 \end{bmatrix} \xRightarrow[R_3-4R_2]{3R_2-R_1} \begin{bmatrix} 3 & 2 & 5 & 1 \\ 0 & -11 & -11 & 11 \\ 0 & 17 & 17 & -17 \end{bmatrix} \xRightarrow[\frac{R_3}{17}+\frac{R_2}{11}]{R_2/(-11)} \begin{bmatrix} 3 & 2 & 5 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $\therefore V_1, V_2, V_3$ do not sp

$\therefore$ Set is linearly dependent and can span R^2 of R^3. Moreo

$V_3 = c_1V_1 + c_2V_2$, where $c_2 = 1$ & $3c_1 + 2c_2 = 5 \therefore$

$\therefore V_3 = V_1 + V_2$

& $V_4 = c_1'V_1 + c_2'V_2$, where $c_2' = -1$, $3c_1' + 2c_2' = 1 \therefore c_1'$

$\therefore V_4 = V_1 - V_2$

$\boxed{\frac{9d}{153}}$ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 2 \\ 3 & 4 & 3 & 1 \end{bmatrix} \underset{\substack{R_2-3R_1 \\ R_3-3R_1}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & -16 \\ 0 & 1 & 0 & -17 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & -17 \\ 0 & 0 & 1 & -16 \end{bmatrix}$ $\therefore V_4$ is linear Combination of V_1, V_2, V_3

and $\det[V_1|V_2|V_3] \neq 0$ $\therefore$ They span R^3

$\boxed{\frac{12}{153}}$

$$\begin{array}{c} \begin{array}{ccccccc} V_1 & V_2 & V_3 & (a) & (b) & (c) & (d) \end{array} \\ \begin{bmatrix} 2 & 3 & -1 & 2 & 0 & 1 & -4 \\ 1 & -1 & 0 & 3 & 0 & 1 & 6 \\ 0 & 5 & 2 & -7 & 0 & 1 & -13 \\ 3 & 2 & 1 & 3 & 0 & 1 & 4 \end{bmatrix} \end{array} \underset{\substack{2R_2-R_1 \\ \Longrightarrow \\ R_4-3R_2}}{\Longrightarrow} \begin{bmatrix} 2 & 3 & -1 & 2 & 0 & 1 & -4 \\ 0 & -5 & 1 & 4 & 0 & 1 & 16 \\ 0 & 5 & 2 & -7 & 0 & 1 & -13 \\ 0 & 5 & 1 & -6 & 0 & -2 & -14 \end{bmatrix} \underset{\substack{R_3+R_2 \\ R_4+R_2}}{\Longrightarrow}$$

$$\begin{bmatrix} 2 & 3 & -1 & 2 & 0 & 1 & -4 \\ 0 & -5 & 1 & 4 & 0 & 1 & 16 \\ 0 & 0 & 3 & -3 & 0 & 2 & 3 \\ 0 & 0 & 2 & -2 & 0 & -1 & 2 \end{bmatrix} \underset{3R_4-2R_3}{\Longrightarrow} \begin{array}{c} \begin{bmatrix} 2 & 3 & -1 & 2 & 0 & 1 & -4 \\ 0 & -5 & 1 & 4 & 0 & 1 & 16 \\ 0 & 0 & 3 & -3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & -7 & 0 \end{bmatrix} \\ \begin{array}{ccccccc} V_1 & V_2 & V_3 & (a) & (b) & (c) & (d) \end{array} \end{array}$$

$\therefore$ (a) It is in $Lin[V_1|V_2|V_3]$

(b) It is in $Lin[V_1|V_2|V_3]$

(c) It is not in $Lin[V_1|V_2|V_3]$ because of inconsistancy in forth component.

(d) It is in $Lin[V_1|V_2|V_3] = 3\langle 2,1,0,3\rangle - 3\langle 3,-1,5,2\rangle + \langle -1,0,2,1\rangle$

$\boxed{\frac{13}{153}}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ $\therefore$ Augmented matrix is $\begin{bmatrix} 1 & 2 & x \\ 1 & 3 & y \\ -1 & 5 & z \end{bmatrix} \underset{\substack{R_1 \\ R_2-R_1 \\ R_2+R_3}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & y-x \\ 0 & 8 & y+z \end{bmatrix}$

$\underset{R_3-8R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & y-x \\ 0 & 0 & y+z-8(y-x) \end{bmatrix}$ $\therefore$ for consistancy $y+z-8(y-x)=0$

$\therefore$ Equation of plane is $8x-7y+z=0$

OR

normal of plane is $\langle 1,1,-1\rangle \times \langle 2,3,5\rangle = \langle 8,-7,1\rangle$

point on plane is $(0,0,0)$ $\therefore$ equation is $8x-7y+z=0$

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$$\begin{bmatrix}3&6&-1&-3\\0&2&3&7\\4&-1&5&8\\1&2&1&3\end{bmatrix}\xrightarrow[R_1-3R_4]{R_3-4R_4}\begin{bmatrix}3&6&-1&-3\\0&2&3&7\\0&-9&1&-4\\0&0&-4&-12\end{bmatrix}\xrightarrow[R_4/(-4)]{9R_2+2R_3}\begin{bmatrix}3&6&-1&-3\\0&2&3&7\\0&0&29&55\\0&0&1&3\end{bmatrix}\xrightarrow{R_3-29R_4}$$

$$\begin{bmatrix}3&6&-1&3\\0&2&-3&7\\0&0&29&55\\0&0&0&-32\end{bmatrix}\quad \therefore \det\left[v_1|v_2|v_3|v_4\right]\neq 0 \quad \therefore \text{They are lin. indep.}$$

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$$\det\begin{bmatrix}3&2&4\\1&-1&0\\1&5&-3\end{bmatrix}=9+0+20+4+6+0=39\neq 0 \quad \therefore \text{They are linearly indep.}$$

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(a) $\cos^2 x+\sin^2 x=1 \quad \therefore 2=2\cos^2 x+2\sin^2 x=2(\cos^2 x)+\frac{1}{2}(4\sin^2 x)$ $\therefore$ Dep.

(b) x, $\cos x$ are lin. indep.

(c) $1, \sin x, \sin 2x$ are lin. indep.

(d) $\cos 2x=\cos^2 x-\sin^2 x \quad \therefore$ Dep.

(e) $(1+x)^2=1+2x+x^2=1(x^2+2x)+\frac{1}{3}(3) \quad \therefore$ lin. dep.

(f) $0, x, x^2$ are lin. dep.

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$$\begin{bmatrix}2&4&2\\-1&2&7\\4&3&-6\end{bmatrix}\xrightarrow[R_3-2R_1]{R_1+2R_2}\begin{bmatrix}2&4&2\\0&8&16\\0&-5&-10\end{bmatrix}\xrightarrow[\frac{R_3}{5}+\frac{R_2}{8}]{R_2/8}\begin{bmatrix}2&4&2\\0&1&2\\0&0&0\end{bmatrix}$$

$\therefore$ The vectors do not span R^3 they rather span R^2 in R^3

$\therefore$ The vectors lie in a plane.

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$$\begin{bmatrix}4&2&-2\\6&3&-3\\8&4&-4\end{bmatrix}\begin{matrix}R_1/2\\R_2/3\\R_3/4\end{matrix}\Rightarrow\begin{bmatrix}2&1&-1\\2&1&-1\\2&1&-1\end{bmatrix}\xrightarrow[R_3-R_1]{R_2-R_1}\begin{bmatrix}2&1&-1\\0&0&0\\0&0&0\end{bmatrix}\quad\therefore \text{The vectors}$$

span R^1 in R^3 $\therefore$ They lie on a line.

$\boxed{\frac{8}{158}}$ For v_1, v_2, v_3 to be dependent $\therefore \det[v_1|v_2|v_3] = 0$

$$\therefore \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^3 - \frac{1}{8} - \frac{1}{8} - \frac{\lambda}{4} - \frac{\lambda}{4} - \frac{\lambda}{4} = \lambda^3 - \frac{3\lambda}{4} - \frac{1}{4} = 0$$

$$\therefore 4\lambda^3 - 3\lambda - 1 = 0 \quad \therefore 3\lambda^3 - 3\lambda + \lambda^3 - 1 = 0$$

$$\therefore 3\lambda(\lambda^2 - 1) + (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$$

$$\therefore 3\lambda(\lambda-1)(\lambda+1) + (\lambda-1)(\lambda^2 + \lambda + 1) = 0$$

$$\therefore (\lambda - 1)[3\lambda(\lambda+1) + \lambda^2 + \lambda + 1] = 0$$

$$\therefore (\lambda-1)(3\lambda^2 + 3\lambda + \lambda^2 + \lambda + 1) = 0$$

$$\therefore (\lambda-1)(4\lambda^2 + 4\lambda + 1) = 0$$

$$\therefore (\lambda-1)(2\lambda+1)^2 = 0$$

$\therefore$ $\lambda = 1$ or $\lambda = -\frac{1}{2}$ gives dependent set of vectors.

$\boxed{\frac{4}{165}}$ (a) $\det\begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{bmatrix} = 0 - 14 - 12 - 2 + 0 + 28 = 0$ $\therefore$ Not a basis for P_2.

(b) $\det\begin{bmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix} = -16 - 2 + 60 - 20 - 6 - 16 = 0$ $\therefore$ Not a basis for P_2.

(c) $\det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 1$ $\therefore$ They are basis for P_2.

(d) $\det\begin{bmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = -20 + 72 + 16 - 120 - 6 + 32 = -26$ $\therefore$ They are basis for P_2

$\boxed{\frac{5}{165}}$ $\det\begin{bmatrix} 3 & 0 & 6 & 1 \\ 3 & -1 & -12 & -1 \\ 6 & -1 & -8 & 0 \\ -6 & 0 & -4 & 2 \end{bmatrix} = \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 + R_3 \end{array} \begin{vmatrix} 3 & 0 & 6 & 1 \\ 0 & 1 & 12 & 2 \\ 0 & -1 & -8 & -2 \\ 0 & -1 & -12 & 2 \end{vmatrix} = \begin{array}{c} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{array} \begin{vmatrix} 3 & 0 & 6 & 1 \\ 0 & 1 & -12 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 3 * 1 * 4 * 4 = 48 \neq 0$

$\therefore$ They are basis for M_{22}.

$\boxed{\frac{12}{165}}$ $\begin{bmatrix}1&1&1\\3&2&-2\\4&3&-1\\6&5&1\end{bmatrix} \xrightarrow[\substack{R_2-3R_1\\R_3-4R_1\\R_4-2R_2}]{} \begin{bmatrix}1&1&1\\0&-1&-5\\0&-1&-5\\0&1&5\end{bmatrix} \xrightarrow[\substack{-R_2\\R_3-R_2\\R_3+R_4}]{} \begin{bmatrix}1&1&1\\0&1&5\\0&0&0\\0&0&0\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}1&0&-4\\0&1&5\\0&0&0\\0&0&0\end{bmatrix}$

$\therefore$ The dimension of the solution space is $3-2 =$ One.

$\left.\begin{matrix}\therefore x-4z=0 & \therefore x=4z\\ \& \ y+5z=0 & \therefore y=-5z\end{matrix}\right\} \therefore \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4z\\-5z\\z\end{pmatrix}=z\begin{pmatrix}4\\-5\\1\end{pmatrix}$

$\therefore$ The basis for solution space is $\langle 4,-5,1\rangle$

$\boxed{\frac{13c}{165}}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2t\\-t\\4t\end{pmatrix}=t\begin{pmatrix}2\\-1\\4\end{pmatrix}$ $\therefore$ The basis is $\langle 2,-1,4\rangle$

$\boxed{\frac{16}{166}}$ $\because \{v_1, v_2, v_3\}$ is a basis for R^3

$\therefore \det\left[v_1 | v_2 | v_3\right] \neq 0$, Consider now the u's

$\therefore \det\left[u_1 | u_2 | u_3\right] = \det\left[v_1 | v_1+v_2 | v_1+v_2+v_3\right] =$

$= \det\left(\left[v_1 | v_2 | v_3\right]\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\right) = \det\left[v_1 | v_2 | v_3\right]\cdot\det\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} =$

$= \det\left[v_1 | v_2 | v_3\right]\cdot 1 \neq 0$

$\therefore \{u_1, u_2, u_3\}$ is a basis for R^3

$\boxed{\frac{3}{174}}$ $\begin{bmatrix}1&2&-1\\2&4&6\\0&0&-8\end{bmatrix} \xrightarrow[2R_1-R_2]{} \begin{bmatrix}1&2&-1\\0&0&-8\\0&0&-8\end{bmatrix} \xrightarrow[\substack{R_2/(-8)\\R_2-R_3}]{} \begin{bmatrix}1&2&-1\\0&0&1\\0&0&0\end{bmatrix}$

(a) $\therefore \{\langle 1,2,-1\rangle \ \& \ \langle 0,0,1\rangle\}$ is a basis for the row space.

$\begin{bmatrix}1&2&0\\2&4&0\\-1&6&-8\end{bmatrix} \xrightarrow[\substack{2R_1-R_2\\R_1+R_3}]{} \begin{bmatrix}1&2&0\\0&0&0\\0&8&-8\end{bmatrix} \xrightarrow{R_3/8} \begin{bmatrix}1&2&0\\0&0&0\\0&1&-1\end{bmatrix}$

(b) $\& \ \{\langle 1,2,0\rangle \ \& \ \langle 0,1,-1\rangle\}$ is a basis for the column space.

(c) $\&$ The rank of the matrix is two.

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$\frac{8}{175}$

$$\begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix} \xrightarrow{\substack{R_2+R_1 \\ R_3-5R_1 \\ R_4+2R_2}} \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ 0 & 1 & -1 & 4 & 11 \\ 0 & 11 & -11 & 2 & 37 \\ 0 & 6 & -6 & 5 & 28 \end{bmatrix} \xrightarrow{\substack{R_3-11R_2 \\ R_4-6R_2}} \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ 0 & 1 & -1 & 4 & 11 \\ 0 & 0 & 0 & -42 & -84 \\ 0 & 0 & 0 & -19 & -38 \end{bmatrix} \xrightarrow{\substack{R_3/(-42) \\ \frac{R_4}{-19}+\frac{R_3}{42}}}$$

$$\begin{array}{ccccc} \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ 0 & 1 & -1 & 4 & -11 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5 \end{array} \Longrightarrow \begin{array}{c} \begin{bmatrix} 1 & -2 & 0 & 4 & -7 \\ 0 & 1 & 4 & -1 & 11 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ V_1 \quad V_2 \quad V_4 \quad V_3 \quad V_5 \end{array}$$

$\therefore$ The space spaned by the five vectors is R^3 of R^4

(a) and the basis is $\boxed{V_1, V_2, V_4}$ (you can change V_3 with V_2 & V_4 with V_5)

(b) to find components of V_3 $\therefore$ $V_3 = c_1V_1 + c_2V_2 + c_3V_4$ $\therefore c_3 = 0$ $\therefore c_2 = -1$

$\therefore c_1 = 4 + 2c_1 = 4 + 2(-1) = 2$ $\quad \therefore \boxed{V_3 = 2V_1 - V_2}$

to find components of V_5 $\therefore V_5 = c_1'V_1 + c_2'V_2 + c_3'V_4$ $\therefore c_3' = 2$ $\therefore c_2' = 11 - 4c_3'$

$= 11 - 8 = 3$ $\therefore c_1' = -7 + 2c_2' = -7 + 2(3) = -1$ $\quad \therefore \boxed{V_5 = -V_1 + 3V_2 + 2V_4}$

$\frac{9}{175}$

$$\begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array}$$
$$\begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{bmatrix} \xrightarrow{\substack{R_2/3 \\ R_1-R_3 \\ R_1-R_4}} \begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 1 & 1 & 1 \\ 0 & -10 & -10 & -10 \\ 0 & -4 & -4 & -4 \end{bmatrix} \xrightarrow{\substack{10R_2+R_3 \\ 4R_2+R_4}}$$

$$\begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ V_1 and one of V_2, V_3, V_4 are Linearly independent.

For V_1 & V_3 being Linearly indep.

$\therefore$ $V_4 = c_1V_1 + c_2V_3$ $\Longrightarrow$ $c_2 = 1$ & $c_1 - c_2 = -5$ $\therefore c_1 = -4$

$\therefore$ $\boxed{V_4 = -4V_1 + V_3}$

& $V_2 = c_3V_1 + c_4V_3$ $\Longrightarrow$ $c_4 = 1$ & $c_3 - c_4 = -3$ $\therefore c_3 = -2$

$\therefore$ $\boxed{V_2 = -2V_1 + V_3}$

$\boxed{\frac{10b}{175}}$ $\begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & -5 \\ 4 & -5 & 9 & 3 \\ 0 & 1 & 1 & -2 \end{bmatrix} \xrightarrow[\substack{R_2-R_1 \\ R_3-2R_1}]{R_1/2} \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 3 & -3 & -13 \\ 0 & 3 & -3 & -13 \\ 0 & 1 & 1 & -2 \end{bmatrix} \xrightarrow[\substack{R_3-R_2 \\ 3R_4-R_2}]{} \begin{array}{cc} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} & \\ \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 3 & -3 & -13 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 7 \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \end{array}$

$\therefore$ The basis for row space is R_1, R_2 & R_4 & for column space is c_1, c_2 & c_3.

$\boxed{\frac{11}{175}}$ ⓐ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$ The entries of diagonal will maximum be 3

$\therefore$ Largest possible value of rank of A is 3.

ⓑ The entries of diagonal will maximum be the same as the smallest of row & column numbers

$\therefore$ Max. rank of A $m \times n$ is the smallest of m, n.

$\boxed{\frac{6}{180}}$ ⓐ $\langle u, v \rangle = u_1 v_1 + u_3 v_3$

axiom (1) $\langle v, u \rangle = v_1 u_1 + v_3 u_3 = u_1 v_1 + u_3 v_3 = \langle u, v \rangle$ $\therefore$ OK

" (2) $\langle u + v, w \rangle = (u_1 + v_1) w_1 + (u_3 + v_3) w_3 = (u_1 w_1 + u_3 w_3) + (v_1 w_1 + v_3 w_3)$
$= \langle u, w \rangle + \langle v, w \rangle$ $\therefore$ OK

axiom (3) $\langle k u, v \rangle = k u_1 v_1 + k u_3 v_3 = k (u_1 v_1 + u_3 v_3) = k \langle u, v \rangle$ $\therefore$ OK

axiom (4) ⓐ $\langle v, v \rangle = v_1 v_1 + v_3 v_3 = v_1^2 + v_3^2$ always ≥ 0 $\therefore$ OK

" (4) ⓑ $\langle v, v \rangle = 0 \implies v_1^2 + v_3^2 = 0$ $\therefore v_1 = v_3 = 0$ $\therefore v = \langle 0, v_2, 0 \rangle \neq 0$

$\therefore$ It is not inner product because axiom 4b fails.

ⓑ $\langle u, v \rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2$

(1) $\langle v, u \rangle = v_1^2 u_1^2 + v_2^2 u_2^2 + v_3^2 u_3^2 = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 = \langle u, v \rangle$ $\therefore$ OK

(2) $\langle u + v, w \rangle = (u_1 + v_1)^2 w_1^2 + (u_2 + v_2)^2 w_2^2 + (u_3 + v_3)^2 w_3^2 = u_1^2 w_1^2 + v_1^2 w_1^2 + 2u_1 v_1 w_1^2 +$
$u_2^2 w_2^2 + v_2^2 w_2^2 + 2u_2 v_2 w_2^2 +$
$u_3^2 w_3^2 + v_3^2 w_3^2 + 2u_3 v_3 w_3^2$
$= \langle u, w \rangle + \langle v, w \rangle + \cdots \neq \langle u, w \rangle + \langle v, w \rangle$

$\therefore$ This axiom fails.

(3) $\langle k u, v \rangle = k^2 u_1^2 v_1^2 + k^2 u_2^2 v_2^2 + k^2 u_3^2 v_3^2 = k^2 \langle u, v \rangle$ $\therefore$ This axiom also fails

(4) ⓐ $\langle v, v \rangle = v_1^2 v_1^2 + v_2^2 v_2^2 + v_3^2 v_3^2 = v_1^4 + v_2^4 + v_3^4 \geq 0$ $\therefore$ OK

(4) ⓑ $\langle v, v \rangle = 0$ $\therefore v_1 = 0, v_2 = 0, v_3 = 0$ $\therefore v = \langle v_1, v_2, v_3 \rangle = \langle 0, 0, 0 \rangle$, $\therefore$ OK

$\therefore$ It is not an inner product because axiom (2) & (3) fail.

(c) $\langle u, v\rangle = 2u_1v_1 + u_2v_2 + 4u_3v_3$

(1) $\langle v, u\rangle = 2v_1u_1 + v_2u_2 + 4v_3u_3 = 2u_1v_1 + u_2v_2 + 4u_3v_3 = \langle u, v\rangle$ ∴ OK

(2) $\langle u+v, w\rangle = 2(u_1+v_1)w_1 + (u_2+v_2)w_2 + 4(u_3+v_3)w_3 = 2u_1w_1 + u_2w_2 + 4u_3w_3 + 2v_1w_1 + v_2w_2 + 4v_3w_3 = \langle u, w\rangle + \langle v, w\rangle$ ∴ OK

(3) $\langle ku, v\rangle = 2ku_1v_1 + ku_2v_2 + 4ku_3v_3 = k(2u_1v_1 + u_2v_2 + 4u_3v_3) = k\langle u, v\rangle$ ∴ OK

(4)ⓐ $\langle v, v\rangle = 2v_1^2 + v_2^2 + 4v_3^2 \geq 0$ ∴ OK

(4)ⓑ $\langle v, v\rangle = 0$ ∴ $v_1 = v_2 = v_3 = 0$ ∴ $v = \langle v_1, v_2, v_3\rangle = \langle 0, 0, 0\rangle$ ∴ OK

∴ This is an inner product space.

Hence:

$\|\langle 1,1,1\rangle\|^2 = 2*1^2 + 1^2 + 4*1^2 = 2+1+4 = 7$

∴ $\|\langle 1,1,1\rangle\| = \sqrt{7}$

(d) $\langle u, v\rangle = u_1v_1 - u_2v_2 + u_3v_3$

(1) $\langle v, u\rangle = v_1u_1 - v_2u_2 + v_3u_3 = u_1v_1 - u_2v_2 + u_3v_3 = \langle u, v\rangle$ ∴ OK

(2) $\langle u+v, w\rangle = (u_1+v_1)w_1 - (u_2+v_2)w_2 + (u_3+v_3)w_3 = u_1w_1 - u_2w_2 + u_3w_3 + v_1w_1 - v_2w_2 + v_3w_3 = \langle u, w\rangle + \langle v, w\rangle$ ∴ OK

(3) $\langle ku, v\rangle = ku_1v_1 - ku_2v_2 + ku_3v_3 = k(u_1v_1 - u_2v_2 + u_3v_3) = k\langle u, v\rangle$ ∴ OK

(4)ⓐ $\langle v, v\rangle = v_1^2 - v_2^2 + v_3^2$ can be negative ∴ This fails

(4)ⓑ $\langle v, v\rangle = 0$ ∴ $v_1^2 + v_3^2 = v_2^2 \Rightarrow v \neq 0$ This also fails

∴ This is not an inner product space because axiom 4 a, b fail

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$\langle u, v\rangle = c_1u_1v_1 + c_2u_2v_2 + c_3u_3v_3$

(1) $\langle v, u\rangle = c_1v_1u_1 + c_2v_2u_2 + c_3v_3u_3 = c_1u_1v_1 + c_2u_2v_2 + c_3u_3v_3 = \langle u, v\rangle$ ∴ OK

(2) $\langle u+v, w\rangle = c_1(u_1+v_1)w_1 + c_2(u_2+v_2)w_2 + c_3(u_3+v_3)w_3 = c_1u_1w_1 + c_2u_2w_2 + c_3u_3w_3 + c_1v_1w_1 + c_2v_2w_2 + c_3v_3w_3 = \langle u, w\rangle + \langle v, w\rangle$ ∴ OK

(3) $\langle ku, v\rangle = c_1ku_1v_1 + c_2ku_2v_2 + c_3ku_3v_3 = k(c_1u_1v_1 + c_2u_2v_2 + c_3u_3v_3) = k\langle u, v\rangle$ ∴ OK

(4)ⓐ $\langle v, v\rangle = c_1v_1^2 + c_2v_2^2 + c_3v_3^2 \geq 0$ because c_1, c_2, c_3 are all > 0 ∴ OK

(4)ⓑ $\langle v, v\rangle = 0$ ∴ $c_1v_1^2 + c_2v_2^2 + c_3v_3^2 = 0$ ∴ $v_1 = v_2 = v_3 = 0$ (because c_1, c_2, c_3 are all > 0)

∴ $v = \langle v_1, v_2, v_3\rangle = \langle 0, 0, 0\rangle$ ∴ OK

∴ This is an inner product space on R^3

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$|w| = \sqrt{\langle w, w\rangle} = \sqrt{a^2w_1^2 + b^2w_2^2 + c^2w_3^2} = \sqrt{a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2} = \sqrt{3}abc.$

16/181 (a) $\langle 1-x+x^2+5x^3, x-3x^2 \rangle = \int_{-1}^{1} (1-x+x^2+5x^3)(x-3x^2)\,dx =$

$= \int_{-1}^{1} (x - x^2 + x^3 + 5x^4 - 3x^2 + 3x^3 - 3x^4 - 15x^5)\,dx = \int_{-1}^{1} (x - 4x^2 + 4x^3 + 2x^4 - 15x^5)\,dx$

$= \frac{x^2}{2} - 4\frac{x^3}{3} + x^4 + 2\frac{x^5}{5} - 15\frac{x^6}{6} \Big|_{-1}^{1} = \cancel{\frac{1-1}{2}}^{0} - \frac{4}{3}(1+1) + \cancel{(1-1)}^{0} + \frac{2}{5}(1+1) - \frac{15}{6}\cancel{(1-1)}^{0} =$

$= -\frac{8}{3} + \frac{4}{5} = \frac{-40+12}{15} = -\frac{28}{15}$

(b) $\langle x-5x^3, 2+8x^2 \rangle = \int_{-1}^{1} (x-5x^3)(2+8x^2)\,dx = \int_{-1}^{1} (2x - 10x^3 + 8x^3 - 40x^5)\,dx$

$= \int_{-1}^{1} (2x - 2x^3 - 40x^5)\,dx = x^2 - \frac{x^4}{2} - \frac{40x^6}{6} \Big|_{-1}^{1} = 0$

5/186 (a) $d(x,y) = \sqrt{\langle x-y, x-y \rangle} = \sqrt{\langle \langle -3,-3 \rangle, \langle -3,-3 \rangle \rangle} = \sqrt{3(-3)(-3) + 2(-3)(-3)} = 3\sqrt{5}$

(b) $d(\langle 3,9 \rangle, \langle 3,9 \rangle) = \sqrt{\langle \langle 0,0 \rangle, \langle 0,0 \rangle \rangle} = \sqrt{3(0)^2 + 2(0)^2} = 0$

9/187 (a) $u \cdot v = 2 - 12 = -10$, $|u| = \sqrt{10}$, $|v| = \sqrt{20}$

$\therefore \cos\theta = \frac{u \cdot v}{|u| \cdot |v|} = \frac{-10}{\sqrt{200}} = -\frac{1}{\sqrt{2}}$

(c) $u \cdot v = -2 + 20 - 18 = 0 \quad \therefore \cos\theta = 0$

(e) $u \cdot v = -3 + 0 - 3 + 0 = -6 \qquad |u| = \sqrt{2}$, $|v| = 3\sqrt{4} = 6$

$\therefore \cos\theta = \frac{u \cdot v}{|u| \cdot |v|} = \frac{-6}{\sqrt{2} \cdot 6} = -\frac{1}{\sqrt{2}}$

11a/187 $A \cdot B = 6 + 12 + 1 - 0 = 19 \longrightarrow |A| = \sqrt{4+36+1+9} = \sqrt{50}$

$|B| = \sqrt{9+4+1+0} = \sqrt{14}$

$\therefore \cos\theta = \frac{A \cdot B}{|A| \cdot |B|} = \frac{19}{\sqrt{50} \cdot \sqrt{14}} = \frac{19}{\sqrt{700}} = \frac{19}{10\sqrt{7}}$

11b/187 $A \cdot B = 2 \cdot (-3) + 4(1) - 1(4) + 3(2) = -6 + 4 - 4 + 6 = 0$

$|A|^2 = 2^2 + 4^2 + (-1)^2 + (3)^2 = 4 + 16 + 1 + 9 = 30 \qquad \therefore |A| = \sqrt{30}$

$|B|^2 = (-3)^2 + (1)^2 + 4^2 + 2^2 = 9 + 1 + 16 + 4 = 30 \qquad \therefore |B| = \sqrt{30}$

$\therefore 0 = \sqrt{30} \cdot \sqrt{30} \cdot \cos\angle_A^B \qquad \therefore \cos\angle_A^B = 0$

$\boxed{\frac{14}{187}}$ (a) $-6+0+0+6=0$ $\therefore$ orthogonal

(b) $2+1-0-3=0$ $\therefore$ "

(c) $0+0+0+0=0$ $\therefore$ "

(d) $4+1-5+6=6$ $\therefore$ Not orthogonal

$\boxed{\frac{15}{187}}$ Let a vector normal to u, v, w be $x = \langle x_1, x_2, x_3, x_4 \rangle$

$\therefore u \cdot x = 0 \quad \therefore 2x_1 + x_2 - 4x_3 = 0$

& $v \cdot x = 0 \quad \therefore -x_1 - x_2 + 2x_3 + 2x_4 = 0$

& $w \cdot x = 0 \quad \therefore 3x_1 + 2x_2 + 5x_3 + 4x_4 = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix} \xrightarrow[R_3+3R_2]{2R_2+R_1}$$

$$\begin{bmatrix} 2 & 1 & -4 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & -1 & 11 & 10 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 2 & 1 & -4 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 11 & 6 \end{bmatrix} \quad \therefore x_3 = \frac{-6}{11} x_4, \; x_2 = 4x_4$$

& $2x_1 + x_2 - 4x_3 = 0 \quad \therefore x_1 = (4x_3 - x_2)/2 \quad \therefore x_1 = \left(-\frac{24}{11}x_4 - 4x_4\right)/2$

$= \frac{x_4}{2} \cdot \left(\frac{-24-44}{11}\right) = \frac{-68}{11} \cdot \frac{x_4}{2} = \frac{-34}{11} x_4$

$\therefore \langle x_1, x_2, x_3, x_4 \rangle = \frac{x_4}{11} \langle -34, 44, -6, 11 \rangle$

$\therefore \frac{x}{|x|} = \frac{\langle -34, 44, -6, 11 \rangle}{\sqrt{34^2+44^2+6^2+11^2}} = \frac{\langle -34, 44, -6, 11 \rangle}{\sqrt{3249}} = \frac{\langle -34, 44, -6, 11 \rangle}{57}$

$\therefore$ The two vectors are $\pm \frac{1}{57} \langle -34, 44, -6, 11 \rangle$.

$\boxed{\frac{26}{197}}$ $u_1 = \frac{1}{3}\langle 2, -2, 1 \rangle, \; u_2 = \frac{1}{3}\langle 2, 1, -2 \rangle, \; u_3 = \frac{1}{3}\langle 1, 2, 2 \rangle$

$u_1 \cdot u_2 = \frac{1}{9}(4-2-2) = 0 \quad \therefore u_1 \perp u_2$

$u_1 \cdot u_3 = \frac{1}{9}(2-4+2) = 0 \quad \therefore u_1 \perp u_3$

$u_2 \cdot u_3 = \frac{1}{9}(2+2-4) = 0 \quad \therefore u_2 \perp u_3$

$\therefore u_1, u_2, u_3$ are orthogonal

$|u_1| = \sqrt{\frac{1}{9}(4+4+1)} = 1, \; |u_2| = \sqrt{\frac{1}{9}(4+1+4)} = 1, \; |u_3| = \sqrt{\frac{1}{9}(1+4+4)} =$

$\therefore u_1, u_2, u_3$ are orthonormal.

$\boxed{\frac{6}{197}}$

$u_1 \cdot u_2 = -1+0+0+1=0 \quad \therefore u_1 \perp u_2$

$u_1 \cdot u_3 = 2+0+0-2=0 \quad \therefore u_1 \perp u_3$

$u_1 \cdot u_4 = -1+0+0+1=0 \quad \therefore u_1 \perp u_4 \quad \therefore u_1 \perp \{u_2, u_3 \,\&\, u_4\}$

$u_2 \cdot u_3 = -2+0+4-2=0 \quad \therefore u_2 \perp u_3$

$u_2 \cdot u_4 = 1+0-2+1=0 \quad \therefore u_2 \perp u_4 \quad \therefore u_2 \perp \{u_1, u_3 \,\&\, u_4\}$

$u_3 \cdot u_4 = -2+6-2-2=0 \quad \therefore u_3 \perp u_4 \quad \therefore u_3 \perp \{u_1, u_2 \,\&\, u_4\}$

$\therefore$ The set of u_1, u_2, u_3 & u_4 is orthogonal in R^4.

To find orthonormal divid by modulus

$\therefore \hat{u}_1, \hat{u}_2, \hat{u}_3$ & $\hat{u}_4$ are $\frac{<1,0,0,1>}{\sqrt{2}}, \frac{<-1,0,2,1>}{\sqrt{6}}, \frac{<2,3,2,-2>}{\sqrt{21}}$ & $\frac{<-1,2,-1,1>}{\sqrt{7}}$ and represent the orthonormal set.

$\boxed{\frac{9}{198}}$

$V_1 = \frac{u_1}{|u_1|} = \frac{<0,2,1,0>}{\sqrt{5}}$

$u_2 = c_1 V_1 + c_2 V_2$ with $c_1 = V_1 \cdot u_2 = \frac{<0,2,1,0>}{\sqrt{5}} \cdot <1,-1,0,0>$

$= \frac{-2}{\sqrt{5}} \quad \therefore u_2 = \frac{-2}{\sqrt{5}} V_1 + c_2 V_2 \quad \therefore c_2 V_2 = u_2 + \frac{2}{\sqrt{5}} V_1 =$

$= <1,-1,0,0> + \frac{2}{\sqrt{5}} \frac{1}{\sqrt{5}} <0,2,1,0> = <1, \frac{-5+4}{5}, \frac{2}{5}, 0> = \frac{1}{5}<5,-1,2,0>$

$\therefore V_2 = \frac{<5,-1,2,0>}{\sqrt{30}}$

$u_3 = c_1' V_1 + c_2' V_2 + c_3' V_3'$ with $c_1' = u_3 \cdot V_1 = <1,2,0,-1> \cdot \frac{<0,2,1,0>}{\sqrt{5}} =$

$= \frac{4}{\sqrt{5}}, \quad c_2' = u_3 \cdot V_2 = <1,2,0,-1> \cdot \frac{<5,-1,2,0>}{\sqrt{30}} = \frac{5-2}{\sqrt{30}} = \frac{3}{\sqrt{30}}$

$\therefore u_3 = \frac{4}{\sqrt{5}} V_1 + \frac{3}{\sqrt{30}} V_2 + c_3' V_3' \Rightarrow c_3' V_3 = u_3 - \frac{4}{\sqrt{5}} V_1 - \frac{3}{\sqrt{30}} V_2 = <1,2,0,-1> +$

$-\frac{4}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} <0,2,1,0> - \frac{3}{\sqrt{30}} \cdot \frac{1}{\sqrt{30}} <5,-1,2,0> = <1 - \frac{15}{30}, 2 - \frac{8}{5} + \frac{3}{30}, \frac{-4}{5} - \frac{6}{30}, -1>$

$= \frac{1}{30} <15, 60-48+3, -24-6, -30> = \frac{1}{30} <15,15,-30,-30> = \frac{1}{2}<1,1,-2,-2>$

$\therefore V_3 = \frac{<1,1,-2,-2>}{\sqrt{10}}$

$u_4 = c_1'' V_1 + c_2'' V_2 + c_3'' V_3 + c_4'' V_4$ with $c_1'' = u_4 \cdot V_1 = <1,0,0,1> \cdot \frac{<0,2,1,0>}{\sqrt{5}} =$

$= \frac{0}{\sqrt{5}} = 0, \quad c_2'' = u_4 \cdot V_2 = <1,0,0,1> \cdot \frac{<5,-1,2,0>}{\sqrt{30}} = \frac{5}{\sqrt{30}}, \quad c_3'' = u_4 \cdot V_3 =$

$= \langle 1,0,0,1\rangle \cdot \frac{\langle 1,1,-2,-2\rangle}{\sqrt{10}} = \frac{1-2}{\sqrt{10}} = -\frac{1}{\sqrt{10}}$

$\therefore u_4 = 0v_1 + \frac{5}{\sqrt{30}}v_2 - \frac{1}{\sqrt{10}}v_3 + c_4'' v_4 \quad \therefore c_4'' v_4 = u_4 - \frac{5}{\sqrt{30}}v_2 + \frac{1}{\sqrt{10}}v_3 =$

$= \langle 1,0,0,1\rangle - \frac{5}{\sqrt{30}}\cdot\frac{1}{\sqrt{30}}\langle 5,-1,2,0\rangle + \frac{1}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}}\cdot\langle 1,1,-2,-2\rangle =$

$= \langle 1-\frac{25}{30}+\frac{1}{10}, \frac{5}{30}+\frac{1}{10}, \frac{-10}{30}-\frac{2}{10}, 1-\frac{2}{10}\rangle = \frac{1}{30}\langle 30-25+3, 5+3, -10-6, 30-6\rangle$

$= \frac{1}{30}\langle 8,8,-16,24\rangle = \frac{8}{30}\langle 1,1,-2,3\rangle$

$\therefore v_4 = \frac{\langle 1,1,-2,3\rangle}{\sqrt{15}}$.

$\therefore$ The orthonormal basis vectors are:

$v_1 = \frac{1}{\sqrt{5}}\langle 0,2,1,0\rangle,\ v_2 = \frac{1}{\sqrt{30}}\langle 5,-1,2,0\rangle,\ v_3 = \frac{1}{\sqrt{10}}\langle 1,1,-2,-2\rangle$ & $v_4 = \frac{1}{\sqrt{15}}\langle 1,1,-2,3\rangle$

$\boxed{\frac{11}{198}}$ $v_1 = \frac{u_1}{|u_1|} = \frac{\langle 1,1,1\rangle}{\sqrt{1^2+2*1^2+3*1^2}} = \frac{\langle 1,1,1\rangle}{\sqrt{6}}$

$u_2 = c_1v_1 + c_2v_2$, with $c_1 = \langle u_2, v_1\rangle = \langle\langle 1,1,0\rangle, \frac{\langle 1,1,1\rangle}{\sqrt{6}}\rangle =$

$= \frac{1}{\sqrt{6}}\cdot(1*1 + 2*1*1 + 3*0*1) = \frac{3}{\sqrt{6}} \quad \therefore u_2 = \frac{3}{\sqrt{6}}\cdot v_1 + c_2v_2$

$\therefore c_2v_2 = u_2 - \frac{3}{\sqrt{6}}v_1 = \langle 1,1,0\rangle - \frac{3}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}\langle 1,1,1\rangle = \langle 1-\frac{3}{6}, 1-\frac{3}{6}, \frac{-3}{6}\rangle$

$= \langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{2}\langle 1,1,-1\rangle$

$\therefore v_2 = \frac{\langle 1,1,-1\rangle}{\sqrt{1^2+2*1^2+3*1^2}} = \frac{\langle 1,1,-1\rangle}{\sqrt{6}}$

$u_3 = c_1'v_1 + c_2'v_2 + c_3'v_3$ & $c_1' = \langle u_3, v_1\rangle = \langle\langle 1,0,0\rangle, \frac{\langle 1,1,1\rangle}{\sqrt{6}}\rangle =$

$= \frac{1}{\sqrt{6}}$, $c_2' = \langle u_3, v_2\rangle = \langle\langle 1,0,0\rangle, \frac{\langle 1,1,-1\rangle}{\sqrt{6}}\rangle = \frac{1}{\sqrt{6}}(1) = \frac{1}{\sqrt{6}}$

$\therefore u_3 = \frac{1}{\sqrt{6}}v_1 + \frac{1}{\sqrt{6}}v_2 + c_3'v_3 \quad \therefore c_3'v_3 = u_3 - \frac{1}{\sqrt{6}}(v_1+v_2) = \langle 1,0,0\rangle +$

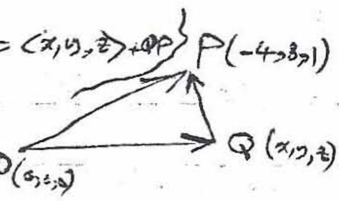
$-\frac{1}{\sqrt{6}}\left(\frac{\langle 1,1,1\rangle}{\sqrt{6}} + \frac{\langle 1,1,-1\rangle}{\sqrt{6}}\right) = \langle 1,0,0\rangle - \frac{1}{6}\langle 2,2,0\rangle = \langle \frac{2}{3}, -\frac{1}{3}, 0\rangle$

$= \frac{1}{3}\langle 2,-1,0\rangle \quad \therefore v_3 = \frac{\langle 2,-1,0\rangle}{\sqrt{2^2+2*1^2}} = \frac{\langle 2,-1,0\rangle}{\sqrt{6}}$

$\therefore v_3 = \frac{\langle 2,-1,0\rangle}{\sqrt{6}}$

$\therefore$ The vectors of the orthonormal basis are: $\frac{\langle 1,1,1\rangle}{\sqrt{6}}, \frac{\langle 1,1,-1\rangle}{\sqrt{6}}$ & $\frac{\langle 2,-1,0\rangle}{\sqrt{6}}$

$\boxed{\frac{23}{199}}$ Vector $OP = \langle -4, 8, 1\rangle$ $\therefore OP = OQ + QP = \langle x, y, z\rangle + QP$



$\therefore OP = t\langle 2, -1, 4\rangle + QP$

we want QP to be minimum, $\therefore OQ \perp QP$

$\therefore \langle 2, -1, 4\rangle \perp QP$

$\therefore \langle 2, -1, 4\rangle \cdot OP = \langle 2, -1, 4\rangle \cdot t\langle 2, -1, 4\rangle + 0$

$\therefore \langle 2, -1, 4\rangle \cdot \langle -4, 8, 1\rangle = t(4+1+16) = 21t$

$\therefore -8 - 8 + 4 = 21t \quad \therefore t = \frac{-12}{21} = \frac{-4}{7}$

$\therefore Q$ is $(x, y, z) = (2t, -t, 4t) = \left(-\frac{8}{7}, \frac{4}{7}, \frac{-16}{7}\right)$

$\boxed{\frac{8}{215}}$ (a) $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ & $B' = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$

$\therefore B\, w_B = B' w_{B'} \qquad \therefore w_B = B^{-1} B' w_{B'} = T_{B' \to B} \cdot w_{B'}$

$\therefore T_{B' \to B} = B^{-1} B' = I^{-1} \cdot B' = I \cdot B' = B' = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, $B' = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ $\therefore B'v' = Bv \quad \therefore v' = B'^{-1} B v$

$\therefore$ Transition matrix from B to B' is $B'^{-1} B = \frac{\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}}{8+3} \cdot I = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$

(c) $B'v' = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \quad \therefore v' = B'^{-1}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -3 \\ -13 \end{bmatrix} = \begin{bmatrix} -3/11 \\ -13/11 \end{bmatrix}$.

(d) $B' w_{B'} = I\, w = w \quad \therefore w_{B'} = B'^{-1} \cdot w = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -5 \end{bmatrix} =$

$= \frac{1}{8+3} \cdot \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \frac{1}{11} \cdot \begin{bmatrix} 12 - 15 \\ -3 - 10 \end{bmatrix} = \begin{bmatrix} -3/11 \\ -13/11 \end{bmatrix}$

$\therefore w_{B'} = \langle -3, -13\rangle / 11$

$\therefore w_B = T\, w_{B'} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -3/11 \\ -13/11 \end{bmatrix} = \begin{bmatrix} (-6+39)/11 \\ (-3-52)/11 \end{bmatrix} = \begin{bmatrix} 33/11 \\ -55/11 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

OR $B w_B = I \cdot w \quad \therefore I \cdot w_B = I \cdot w \quad \therefore w_B = w = \langle 3, -5\rangle$

9/216 $B = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}$ & $B' = \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$

(a) $\therefore B V_B = B' V_{B'}$ $\therefore V_B = B^{-1} B' V_{B'} = T_{B' \to B} \cdot V_{B'}$

$\therefore T_{B' \to B} = B^{-1} B' = \frac{1}{-2-8} \begin{bmatrix} -1 & -4 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix} = \frac{1}{-10} \begin{bmatrix} -13 & 5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1.3 & -.5 \\ -.4 & 0 \end{bmatrix}$

(b) $T_{B \to B'} = \frac{1}{T_{B' \to B}} = \frac{1}{0 - .2} \begin{bmatrix} 0 & .5 \\ .4 & 1.3 \end{bmatrix} = -5 \begin{bmatrix} 0 & .5 \\ .4 & 1.3 \end{bmatrix} = \begin{bmatrix} 0 & -2.5 \\ -2 & -6.5 \end{bmatrix}$

(c) $w = B w_B$ $\therefore w_B = B^{-1} w = \begin{bmatrix} -1 & -4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} / (-10) = \begin{bmatrix} 17 \\ -16 \end{bmatrix} / (-10) = \begin{bmatrix} -1.7 \\ +1.6 \end{bmatrix}$

& $w_{B'} = T_{B \to B'} \cdot w_B = \begin{bmatrix} 0 & -2.5 \\ -2 & -6.5 \end{bmatrix} \begin{bmatrix} -1.7 \\ +1.6 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$

(d) $w = B' w_{B'} \Rightarrow w_{B'} = (B')^{-1} \cdot w = \frac{1}{-1+3} \cdot \begin{bmatrix} -1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -8 \\ -14 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$

(Note: $w_{B'}$ is obtained by two consistent methods).

10/216 (a) $B V_B = B' V_{B'}$ $\therefore V_{B'} = (B')^{-1} B V_B$

$\therefore$ To change V_B to V_B' we multiply it by $[B']^{-1} B = \begin{bmatrix} -6 & -2 & -2 \\ -6 & -6 & -3 \\ 0 & 4 & 7 \end{bmatrix}$.

$\begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 6 \\ -3 & -1 & -1 \end{bmatrix} = \dfrac{\begin{bmatrix} -30 & 42 & -24 \\ 6 & -42 & 24 \\ -6 & -6 & 24 \end{bmatrix}^T \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 6 \\ -3 & -1 & -1 \end{bmatrix}}{36*7+12*4-12*7-18*4} = \frac{1}{144} \begin{bmatrix} -30 & 6 & -6 \\ 42 & -42 & -6 \\ -24 & 24 & 24 \end{bmatrix} \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 6 \\ -3 & -1 & -1 \end{bmatrix} =$

$= \frac{1}{24} \begin{bmatrix} -5 & 1 & -1 \\ 7 & -7 & -1 \\ -4 & 4 & 4 \end{bmatrix} \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 6 \\ -3 & -1 & -1 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 18 & 18 & 2 \\ -18 & -34 & -34 \\ 0 & 16 & 16 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 9 & 9 & 1 \\ -9 & -17 & -17 \\ 0 & 8 & 8 \end{bmatrix}$

(b) $B w_B = w_{standard}$ $\therefore \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 6 \\ -3 & -1 & -1 \end{bmatrix} w_B = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$ $\therefore \Delta = 6+54+6-18 = 48$, $\Delta_{c_1} = 10+90-8+10-24-30 = 48$

$\Delta_{c_2} = 24+90+24-90 = 48$, $\Delta_{c_3} = 30+72-30-24 = 48$

$\therefore w_B = \langle c_1, c_2, c_3 \rangle = \langle \frac{48}{48}, \frac{48}{48}, \frac{48}{48} \rangle = \langle 1, 1, 1 \rangle$

& $w_{standard} = B' w_B'$ or $B' w_B' = w_{standard}$, to solve for w_B', the augmented matrix is

$\therefore \left[\begin{array}{ccc|c} -6 & -2 & -2 & -5 \\ -6 & -6 & -3 & 8 \\ 0 & 4 & 7 & -5 \end{array}\right] \xrightarrow[R_2 - R_1]{R_1/(-2)} \left[\begin{array}{ccc|c} 3 & 1 & 1 & 5/2 \\ 0 & -4 & -1 & 13 \\ 0 & 4 & 7 & -5 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 3 & 1 & 1 & 5/2 \\ 0 & -4 & -1 & 13 \\ 0 & 0 & 6 & 8 \end{array}\right]$

$\therefore c_3 = \frac{8}{6} = 4/3$

$-4c_2 = 13 + c_3 = 13 + \frac{4}{3} = \frac{43}{3}$

$\therefore c_2 = \frac{43}{-12}$

$3c_1 = \frac{5}{2} - c_2 - c_3 = \frac{5}{2} + \frac{43}{12} - \frac{4}{3} = \frac{30+43-16}{12} = \frac{57}{12}$

$\therefore c_1 = \frac{19}{12}$

$\therefore w_{B'} = \langle \frac{19}{12}, \frac{-43}{12}, \frac{4}{3} \rangle$

(c) $w_{B'} = \frac{1}{12} \begin{bmatrix} 9 & 9 & 1 \\ -9 & -17 & -17 \\ 0 & 8 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 9+9+1 \\ -9-17-17 \\ 0+8+8 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 19 \\ -43 \\ 16 \end{bmatrix} = \begin{bmatrix} 19/12 \\ -43/12 \\ 4/3 \end{bmatrix} = \langle \frac{19}{12}, \frac{-43}{12}, \frac{4}{3} \rangle$ ok.

$\boxed{\frac{11}{216}}$

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \det B = -2+4+1+1-2-4 = -2 \neq 0$$

$\therefore$ B is a basis for $\mathbb{R}^3$

$$B' = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{bmatrix} \Rightarrow \det B' = 6+0+3-5-2-0 = 2 \neq 0$$

$\therefore$ B' is also a basis for $\mathbb{R}^3$

$$T_{B\to B'} = B'^{-1}B = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 1 & -1 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & 4 & 5 \\ -4 & -6 & -1 \\ 10 & 2 & 12 \end{bmatrix}$$

$$w = Bw_B \Rightarrow w_B = B^{-1}w = \frac{1}{-2}\begin{bmatrix} -3 & -1 & 5 \\ 1 & 1 & -3 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -18 \\ 18 \\ 10 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$$

$\therefore$ $w_B = \langle 9, -9, -5 \rangle$

$$w_{B'} = B'^{-1}w = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 1 & -1 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -7 \\ +23 \\ +12 \end{bmatrix}$$

$\therefore$ $w_{B'} = +\frac{1}{2}\langle -7, 23, 12 \rangle$

$w_{B'}$ also $= T_{B\to B'} \cdot w_B = \frac{1}{2}\begin{bmatrix} 6 & 4 & 5 \\ -4 & -6 & -1 \\ 10 & 2 & 12 \end{bmatrix}\begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -7 \\ 23 \\ 12 \end{bmatrix}$ $\therefore$ OK

$u_1 = \langle 1, 0 \rangle$, $u_2 = \langle 1, 1 \rangle \in R^2$

$\because \det[u_1 | u_2] = \det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$ $\quad \therefore \{u_1, u_2\}$ are Linearly Independent

$\therefore \{u_1, u_2\}$ can be a basis for R^2

$\because V = \langle 1, -2 \rangle = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = B_u \cdot V_u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} [V_u]$

$\therefore V_u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \dfrac{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}{1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\therefore V_u = \langle 3, -2 \rangle$

$\langle w, w' \rangle = 5 w_1 w_1' + 12 w_2 w_2'$

axiom one OK, two OK, three OK, axiom four:

$\langle w, w \rangle = 5 w_1^2 + 12 w_2^2 > 0$ $\quad \therefore$ OK

$\langle w, w \rangle = 0$ iff $w_1 = w_2 = 0 \Rightarrow w = \langle 0, 0 \rangle$ $\quad \therefore$ OK

$\therefore$ It is an inner product for R^2

$V_1 = \dfrac{\langle 1, 0 \rangle}{|\langle 1, 0 \rangle|} = \dfrac{\langle 1, 0 \rangle}{\sqrt{5(1)^2 + 12(0)^2}} = \dfrac{1}{\sqrt{5}} \langle 1, 0 \rangle$

$u_2 = c_1 V_1 + c_2 V_2$ $\quad$ where $c_1 = \langle u_2, V_1 \rangle = \dfrac{1}{\sqrt{5}} \left(5(1)(1) + 12(1)(0) \right)$

$= \dfrac{1}{\sqrt{5}} (5 + 0) = \dfrac{5}{\sqrt{5}} = \sqrt{5}$

$\therefore c_2 V_2 = u_2 - c_1 V_1 = \langle 1, 1 \rangle - \sqrt{5} \cdot \dfrac{1}{\sqrt{5}} \langle 1, 0 \rangle = \langle 0, 1 \rangle$

$\therefore V_2 = \dfrac{\langle 0, 1 \rangle}{|\langle 0, 1 \rangle|} = \dfrac{\langle 0, 1 \rangle}{\sqrt{5(0)^2 + 12(1)^2}} = \dfrac{\langle 0, 1 \rangle}{\sqrt{12}} = \dfrac{1}{\sqrt{12}} \langle 0, 1 \rangle$

$\therefore$ The orthonormal basis is

$V_1 = \dfrac{1}{\sqrt{5}} \langle 1, 0 \rangle$, $V_2 = \dfrac{1}{\sqrt{12}} \langle 0, 1 \rangle$

$\therefore V = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = B_v V_v = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{12}} \end{bmatrix} V_v \Rightarrow V_v = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{12}} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \sqrt{60} \begin{bmatrix} \frac{1}{\sqrt{12}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$= \sqrt{60} \begin{bmatrix} \frac{1}{\sqrt{12}} \\ -2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ -\cdots \end{bmatrix}$

$\therefore V_v = \langle \sqrt{5}, -2\sqrt{12} \rangle$

12/216 (a) $B=\begin{bmatrix}6 & 10\\ 3 & 2\end{bmatrix}$, $B'=\begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix}$ $\therefore$ $BV_B = B'V_{B'}$ $\therefore V_B = B^{-1}B'V_{B'}$

$\therefore$ The transition matrix from $V_{B'}$ to V_B is $B^{-1}B' = \begin{bmatrix}6 & 10\\ 3 & 2\end{bmatrix}^{-1}\begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix} =$

$$= \frac{\begin{bmatrix}2 & -10\\ -3 & 6\end{bmatrix}}{12-30}\cdot\begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix} = \frac{1}{-18}\cdot\begin{bmatrix}4 & -14\\ -6 & 3\end{bmatrix} = \begin{bmatrix}-\frac{2}{9} & \frac{7}{9}\\ \frac{1}{3} & -\frac{1}{6}\end{bmatrix}$$

(b) The transition matrix from V_B to $V_{B'}$ is $(B^{-1}B')^{-1} = \left\{-\frac{1}{18}\begin{bmatrix}4 & -14\\ -6 & 3\end{bmatrix}\right\}^{-1} =$

$$= -18\cdot\begin{bmatrix}4 & -14\\ -6 & 3\end{bmatrix}^{-1} = -18\cdot\frac{1}{12-84}\begin{bmatrix}3 & 14\\ 6 & 4\end{bmatrix} = \frac{-18}{-72}\cdot\begin{bmatrix}3 & 14\\ 6 & 4\end{bmatrix} = \frac{1}{4}\begin{bmatrix}3 & 14\\ 6 & 4\end{bmatrix} =$$

$$= \begin{bmatrix}\frac{3}{4} & \frac{7}{2}\\ \frac{3}{2} & 1\end{bmatrix}$$

(c) $V_{standard} = BV_B = B'V_{B'}$ $\therefore \begin{bmatrix}-4\\ 1\end{bmatrix} = \begin{bmatrix}6 & 10\\ 3 & 2\end{bmatrix}V_B$ $\therefore V_B = \begin{bmatrix}6 & 10\\ 3 & 2\end{bmatrix}^{-1}\begin{bmatrix}-4\\ 1\end{bmatrix} =$

$$= -\frac{1}{18}\begin{bmatrix}2 & -10\\ -3 & 6\end{bmatrix}\begin{bmatrix}-4\\ 1\end{bmatrix} = -\frac{1}{18}\begin{bmatrix}-18\\ +18\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix} \quad \therefore V_B = P_B = \langle 1, -1\rangle$$

& $V_{B'} = \begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix}^{-1}\begin{bmatrix}-4\\ 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2 & -3\\ 0 & 2\end{bmatrix}\begin{bmatrix}-4\\ 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-11\\ 2\end{bmatrix} = \begin{bmatrix}-\frac{11}{4}\\ \frac{1}{2}\end{bmatrix}$ $\therefore V_B' = P_{B'} = \langle -\frac{11}{4}, \frac{1}{2}\rangle$

(d) $V_{B'} = \begin{bmatrix}3/4 & 7/2\\ 3/2 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} = \begin{bmatrix}-11/4\\ \frac{1}{2}\end{bmatrix}$ $\therefore V_{B'} = P_{B'} = \langle -\frac{11}{4}, \frac{1}{2}\rangle$

19e/217 If rows or columns are $\perp$ $\therefore$ orthogonal provided norms are 1

$R_1\cdot R_2 = \frac{1}{4} - \frac{5}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3-5+1+1}{12} = 0$ $\therefore R_1 \perp R_2$

$R_1\cdot R_3 = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} - \frac{5}{12} = 0$ $\therefore R_1 \perp R_3$

$R_1\cdot R_4 = \frac{1}{4} + \frac{1}{12} - \frac{5}{12} + \frac{1}{12} = 0$ $\therefore R_1 \perp R_4$

$R_2\cdot R_3 = \frac{1}{4} - \frac{5}{36} + \frac{1}{36} - \frac{5}{36} = \frac{9-5+1-5}{36} = 0$ $\therefore R_2 \perp R_3$

$R_2\cdot R_4 = \frac{1}{4} - \frac{5}{36} - \frac{5}{36} + \frac{1}{36} = 0$ $\therefore R_2 \perp R_4$

$R_3\cdot R_4 = \frac{1}{4} + \frac{1}{36} - \frac{5}{36} - \frac{5}{36} = 0$ $\therefore R_3 \perp R_4$

$\therefore$ The matrix is orthogonal provided $|R_1| = |R_2| = |R_3| = |R_4| = 1$

& since $R_1 = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 1$, $R_2 = \sqrt{\frac{1}{4}+\frac{25}{36}+\frac{1}{36}+\frac{1}{36}} = 1$, $R_3 = \sqrt{\frac{1}{4}+\frac{1}{36}+\frac{1}{36}+\frac{25}{36}} = 1$, $R_4 = \sqrt{\frac{1}{4}+\frac{1}{36}+\frac{25}{36}+\frac{1}{36}} = 1$

$\therefore$ It is orthogonal.

20e / 217 since for orthogonal matrix $A^T = A^{-1}$ and since A is symmetric then $A^T = A$, $A^T = A^{-1}$ $\therefore A = A^{-1}$ $\therefore$ The inverse is the same as the matrix.

OR:

$$\left[\begin{array}{cccc|cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -5/6 & 1/6 & 1/6 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 1/6 & 1/6 & -5/6 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 1/6 & -5/6 & 1/6 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} 2R_1 \\ 6R_2 \\ 6R_3 \\ 6R_4 \end{array} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 3 & -5 & 1 & 1 & 0 & 6 & 0 & 0 \\ 3 & 1 & 1 & -5 & 0 & 0 & 6 & 0 \\ 3 & 1 & -5 & 1 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{array}{l} \Longrightarrow \\ R_2-3R_1 \\ R_3-R_2 \\ R_4-R_3 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & -8 & -2 & -2 & -6 & 6 & 0 & 0 \\ 0 & +6 & 0 & -6 & 0 & -6 & 6 & 0 \\ 0 & 0 & -6 & 6 & 0 & 0 & -6 & 6 \end{bmatrix}$$

$$\begin{array}{l} \Longrightarrow \\ R_2/(-2) \\ R_3/(+6) \\ R_4/(-6) \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 1 & 3 & -3 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & +1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{array}{l} \Longrightarrow \\ \\ R_2-4R_3 \\ \Longrightarrow \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 1 & 3 & -3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 3 & 1 & -4 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{array}{l} \\ \Longrightarrow \\ \\ R_4-R_3 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 1 & 3 & -3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 3 & 1 & -4 & 0 \\ 0 & 0 & 0 & -6 & -3 & -1 & 5 & -1 \end{bmatrix}$$

$$\begin{array}{l} \\ \Longrightarrow \\ \\ R_4/(-6) \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 12/6 & 0 & 0 & 0 \\ 0 & 4 & 1 & 1 & 18/6 & -18/6 & 0 & 0 \\ 0 & 0 & 1 & 5 & 18/6 & 6/6 & -24/6 & 0 \\ 0 & 0 & 0 & 1 & \frac{3}{6} & \frac{1}{6} & -5/6 & \frac{1}{6} \end{bmatrix} \begin{array}{l} R_1-R_4 \\ R_2-R_4 \\ R_3-5R_4 \\ \Longrightarrow \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 & 9/6 & -1/6 & 5/6 & -1/6 \\ 0 & 4 & 1 & 0 & 15/6 & -19/6 & 5/6 & -1/6 \\ 0 & 0 & 1 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ 0 & 0 & 0 & 1 & \frac{3}{6} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{array}{l} R_1-R_3 \\ R_2-R_3 \\ \Longrightarrow \\ \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 6/6 & -2/6 & 4/6 & 4/6 \\ 0 & 4 & 0 & 0 & 12/6 & -24/6 & 4/6 & 4/6 \\ 0 & 0 & 1 & 0 & 3/6 & 1/6 & 1/6 & -5/6 \\ 0 & 0 & 0 & 1 & 3/6 & 1/6 & -5/6 & 1/6 \end{bmatrix}$$

$$\begin{array}{l} 4R_1-R_2 \\ R_2/4 \\ \Longrightarrow \\ \end{array} \begin{bmatrix} 4 & 0 & 0 & 0 & \frac{12}{6} & \frac{12}{6} & \frac{12}{6} & \frac{12}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{array}{l} R_1/4 \\ \\ \Longrightarrow \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 & -5/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 0 & 1/2 & 1/6 & 1/6 & -5/6 \\ 0 & 0 & 0 & 1 & 1/2 & 1/6 & -5/6 & 1/6 \end{bmatrix} \therefore A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ \frac{1}{2} & -5/6 & 1/6 & 1/6 \\ \frac{1}{2} & 1/6 & 1/6 & -5/6 \\ \frac{1}{2} & 1/6 & -5/6 & 1/6 \end{bmatrix}$$

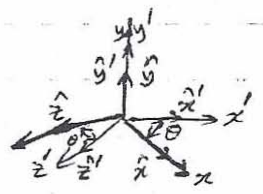
27 / 218 $V = \begin{bmatrix} 4/5 & -3/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} V'$, since the given matrix is orthogonal $\therefore A^T = A^{-1}$

$$\therefore V' = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} V$$

$\therefore$ (a) $V' = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 12/5 \\ -9/5 \\ -7 \end{bmatrix}$, (b) $V' = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 10/5 \\ 5/5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$

(c) $V' = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -9 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} -42/5 \\ 19/5 \\ -3 \end{bmatrix}$, (d) $V' = A \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

31 / 218 (a) Analizing $\therefore \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} (\cos\theta\,\hat{x}' + \sin\theta\,\hat{z}') \\ \hat{y}' \\ (\cos\theta\,\hat{z}' - \sin\theta\,\hat{x}') \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}$



$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \cdot \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}^T \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}$$

$$x\hat{x} + y\hat{y} + z\hat{z} = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$$

$$\left[\hat{x} \,|\, \hat{y} \,|\, \hat{z}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\hat{x}' \,|\, \hat{y}' \,|\, \hat{z}'\right] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \cdot \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}^T \Rightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \therefore A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ +\sin\theta & 0 & \cos\theta \end{bmatrix}$$

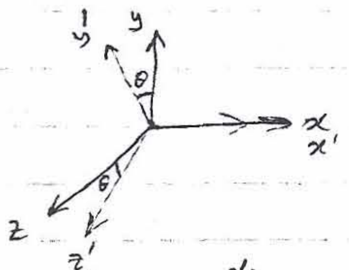
OR $\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}^T \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}$

(b) $x\hat{x}+y\hat{y}+z\hat{z} = x'\hat{x}'+y'\hat{y}'+z'\hat{z}'$

but $\hat{x} = \hat{x}'$

& $\hat{y} = \cos\theta\cdot\hat{y}' - \sin\theta\cdot\hat{z}'$

& $\hat{z} = \cos\theta\cdot\hat{z}' + \sin\theta\,\hat{y}'$



$\therefore\ x\hat{x}+y\hat{y}+z\hat{z} = x\hat{x}' + y(\cos\theta\,\hat{y}' - \sin\theta\,\hat{z}') + z(\cos\theta\,\hat{z}' + \sin\theta\,\hat{y}')$

$= x\hat{x}' + (y\cos\theta + z\sin\theta)\hat{y}' + (y\sin\theta + z\cos\theta)\hat{z}' = x'\hat{x}'+y'\hat{y}'+z'\hat{z}'$

$\therefore \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y\cos\theta + z\sin\theta \\ -y\sin\theta + z\cos\theta \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$

$\boxed{\frac{1}{219}}$ (a) $0x + 0y + 0z = 0$ This equation is true for any (x, y, z) $\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$.

(b) $\begin{bmatrix} 2 & -3 & 1 \\ 6 & -9 & 3 \\ -4 & 6 & -2 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-3R_1} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore 2x - 3y + z = 0$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^2$ (plane with equation $2x - 3y + z = 0$).

(c) $\begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & -3 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2+4R_1} \begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 33 \\ 0 & 0 & -11 \end{bmatrix} \xrightarrow[3R_3+R_2]{R_2/33} \begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore z = 0$ & $x - 2y + 7(0) = 0$

$\therefore x = 2y$ let $y = t$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^1$ (line with equation $x = 2t,\ y = t,\ z = 0$).

(d) $\begin{bmatrix} 1 & 4 & 8 \\ 2 & 5 & 6 \\ 3 & 1 & -4 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 4 & 8 \\ 0 & -3 & -10 \\ 0 & -11 & -28 \end{bmatrix} \xrightarrow[3R_3-11R_2]{-R_2} \begin{bmatrix} 1 & 4 & 8 \\ 0 & 3 & 10 \\ 0 & 0 & 26 \end{bmatrix}$ $\therefore z = 0 \ \therefore y = 0 \ \therefore x = 0$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (the origin).

$\boxed{\frac{5}{220}}$ Let $u = \langle \sqrt{a_1}, \sqrt{a_2}, \cdots\cdots, \sqrt{a_n} \rangle$, $v = \langle \frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \frac{1}{\sqrt{a_3}} \cdots, \frac{1}{\sqrt{a_n}} \rangle$

$\therefore$ By Cauchy-Schwartz:

$(u\cdot u)(v\cdot v) \geqslant (u\cdot v)^2$

$\therefore (a_1 + a_2 + \cdots a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_n}\right) \geqslant (\underbrace{1 + 1 + \cdots + 1}_{n})^2$

$\therefore (a_1 + a_2 + \cdots a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots \frac{1}{a_n}\right) \geqslant n^2$

The condition for this is ~~that~~ $\frac{1}{\sqrt{a_n}}$ exists i.e $a_n > 0$

$\boxed{\frac{1,2,3}{275}}$ (d) $|A - \lambda I| = \begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) + 7 = -4 + \lambda^2 + 7 = \lambda^2 + 3 = 0$

$\therefore$ A has no real eign values and hence no real eign space.

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(a) $P(\lambda)=\begin{vmatrix}4-\lambda & 0 & 1\\ -2 & 1-\lambda & 0\\ -2 & 0 & 1-\lambda\end{vmatrix} = (4-\lambda)(1-\lambda)^2+2(1-\lambda) = (1-\lambda)[(4-\lambda)(1-\lambda)+2] =$

$= (1-\lambda)(\lambda^2-5\lambda+6) = (1-\lambda)(\lambda-3)(\lambda-2)=0$

$\therefore\ \lambda = 1, 2, 3$ (eign values)

for $\lambda=1$ $\therefore \begin{bmatrix}3&0&1\\-2&0&0\\-2&0&0\end{bmatrix} \xrightarrow[R_3-R_2]{3R_2+2R_1} \begin{bmatrix}3&0&1\\0&0&2\\0&0&0\end{bmatrix}$ $\therefore 2x_3=0$, $3x_1+x_3=0$ $\therefore x_1=0$

$\therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$

$\therefore$ The eign space for $\lambda=1$ is $x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$ and its basis is $\begin{bmatrix}0\\1\\0\end{bmatrix}$.

for $\lambda=2$ $\therefore \begin{bmatrix}2&0&1\\-2&-1&0\\-2&0&-1\end{bmatrix} \xrightarrow[R_1+R_3]{R_1+R_2} \begin{bmatrix}2&0&1\\0&-1&1\\0&0&0\end{bmatrix}$ $\therefore x_3=x_2$, $2x_1=-x_3$ $\therefore$ eign space is $x_1\begin{bmatrix}1\\-2\\-2\end{bmatrix}$ with basis $\begin{bmatrix}1\\-2\\-2\end{bmatrix}$

for $\lambda=3$ $\therefore \begin{bmatrix}1&0&1\\-2&-2&0\\-2&0&-2\end{bmatrix} \xrightarrow[R_3+2R_1]{R_2+2R_1} \begin{bmatrix}1&0&1\\0&-2&2\\0&0&0\end{bmatrix}$ $\therefore x_1=-x_3$, $x_2=x_3$ $\therefore$ eign space is $x_1\begin{bmatrix}1\\-1\\-1\end{bmatrix}$ with basis $\begin{bmatrix}1\\-1\\-1\end{bmatrix}$.

(c) $P(\lambda)=\begin{vmatrix}-2-\lambda&0&1\\-6&-2-\lambda&0\\19&5&-4-\lambda\end{vmatrix} = (4+4\lambda+\lambda^2)(-4-\lambda)-30+38+19\lambda = -16-16\lambda-4\lambda^2-4\lambda-4\lambda^2-\lambda^3$

$+8+19\lambda = -\lambda^3-8\lambda^2-\lambda-8 = -\lambda(\lambda^2+1)-8(\lambda^2+1) = -(\lambda^2+1)(\lambda+8)=0$

$\therefore\ \lambda=-8$ only (eign value)

Its eign space is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ given by $\begin{bmatrix}6&0&1\\-6&6&0\\19&5&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\therefore \begin{bmatrix}6&0&1\\-6&6&0\\19&5&4\end{bmatrix} \xrightarrow[19R_1-6R_3]{R_1+R_2} \begin{bmatrix}6&0&1\\0&6&1\\0&-30&-5\end{bmatrix} \xrightarrow{R_3+5R_2} \begin{bmatrix}6&0&1\\0&6&1\\0&0&0\end{bmatrix}$

$\therefore 6x_2=-x_3$, $6x_1=-x_3$ $\therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_1\\-6x_1\end{bmatrix}=x_1\begin{bmatrix}1\\1\\-6\end{bmatrix}$ with eign basis $\begin{bmatrix}1\\1\\-6\end{bmatrix}$.

(f) $P(\lambda)=\begin{vmatrix}5-\lambda&6&2\\0&-1-\lambda&-8\\1&0&-2-\lambda\end{vmatrix} = (5-\lambda)(\lambda^2+3\lambda+2)-48+2(1+\lambda) = 5\lambda^2+15\lambda+10-\lambda^3-3\lambda^2-2\lambda-48$

$+2+2\lambda = -\lambda^3+2\lambda^2+15\lambda-36=0$ (characteristic equation)

$\therefore P(\lambda)=(\lambda-3)(-\lambda^2-\lambda+12)=(\lambda-3)(4+\lambda)(3-\lambda)=0$

$\therefore$ Eign values are $3, -4$

for $\lambda=3$ $\therefore \begin{bmatrix}2&6&2\\0&-4&-8\\1&0&-5\end{bmatrix} \xrightarrow[R_2/-4,\ 2R_3-R_1]{R_1/2} \begin{bmatrix}1&3&1\\0&1&2\\0&-6&-12\end{bmatrix} \xrightarrow{R_3+6R_2} \begin{bmatrix}1&3&1\\0&1&2\\0&0&0\end{bmatrix}$ $\therefore x_2=-2x_3$, $x_1=-x_3-3x_2$
$=-x_3+6x_3=5x_3$

$\therefore$ for $\lambda=3$ eign space is $x_3\begin{bmatrix}5\\-2\\1\end{bmatrix}$ with basis $\begin{bmatrix}5\\-2\\1\end{bmatrix}$.

for $\lambda=-4$ $\therefore \begin{bmatrix}9&6&2\\0&3&-8\\1&0&2\end{bmatrix} \xrightarrow{9R_3-R_1} \begin{bmatrix}9&6&2\\0&3&-8\\0&-6&16\end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix}9&6&2\\0&3&-8\\0&0&0\end{bmatrix}$ $\therefore 3x_2=8x_3$, $9x_1=-2x_3-6x_2$
$=-2x_3-16x_3=-18x_3$
$\therefore x_1=-2x_3$

$\therefore$ eignspace $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{x_3}{3}\begin{bmatrix}-6\\8\\3\end{bmatrix}$ with basis $\begin{bmatrix}-6\\8\\3\end{bmatrix}$.

$\boxed{\dfrac{8,9,10}{275}}$ (a) $P(\lambda) = \begin{vmatrix} -\lambda & 0 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (1-\lambda)\left[-\lambda^2(2+\lambda)+2+\lambda\right] =$

$= (1-\lambda)(-\lambda^2+1)(2+\lambda) = (1-\lambda)(1-\lambda)(1+\lambda)(2+\lambda) = (1-\lambda)^2(1+\lambda)(2+\lambda) = 0$

∴ The characteristic equation is $(1-\lambda)^2(1+\lambda)(2+\lambda) = 0$

& The eign values are $1, -1, -2$

for $\lambda = 1$ ∴ $\begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1+R_2+R_3]{R_2+R_1} \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ∴ $-x_2+3x_3=0$ ∴ $x_2 = 3x_3$ & $-x_1+2x_3=0$ ∴ $x_1 = 2x_3$

∴ eign space for $\lambda=1$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 3x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ with basis $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

for $\lambda = -1$ ∴ $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_1+R_3-R_2]{R_1-R_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ ∴ $2x_4 = 0$ ∴ $x_4 = 0$, $x_2 = x_3$, $x_1 = -2x_3$

∴ eign space for $\lambda=-1$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ with basis $\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

for $\lambda = -2$ ∴ $\begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[2R_2-R_1]{R_1/2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[4R_3-R_2]{R_2/4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ ∴ $x_4 = x_2 = 0$, $x_1 = -x_3$

∴ eign space for $\lambda = -2$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ with basis $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\dfrac{18}{276}}$ Let λ be eign value of A and λ' be eign value of A^2

∴ $\det|A-\lambda I| = 0$ & $\det|A^2-\lambda' I| = 0$

but $A^2 - \lambda' I = (A-\sqrt{\lambda'}I)(A+\sqrt{\lambda'}I)$ ∴ $|A^2-\lambda' I| = |(A-\sqrt{\lambda'}I)(A+\sqrt{\lambda'}I)| =$

$= |A-\sqrt{\lambda'}I|\cdot|A+\sqrt{\lambda'}I| = 0$ This means that $\pm\sqrt{\lambda'}$ are eign values of A

∴ $\lambda = \pm\sqrt{\lambda'}$ squaring ∴ $\lambda^2 = \lambda'$ ∴ eign value of A^2 is $\lambda' = \lambda^2$.

Let $\lambda' = k^n$ be eign value of A^n ∴ $|A^n - k^n I| = 0$ but $(A^n - k^n I) = (A-kI)(A^{n-1} + A^{n-2}k + A^{n-3}k^2$ $+ A^{n-4}k^3 + \dots + A^2k^{n-3} + Ak^{n-2} + k^{n-1}I)$ ∴ $|A^n - k^n I| = 0 \Rightarrow |A-kI|\cdot|A^{n-1}+A^{n-2}k+\dots+k^{n-1}I| = 0$

∴ either $|A-kI| = 0$ ∴ k is an eign value of A ∴ $k = \lambda$ ∴ $\lambda' = k^n = \lambda^n$ is eign value of A^n.

$\frac{1}{283}$ $A=\begin{bmatrix}2&0\\1&2\end{bmatrix}$ $\therefore |A-\lambda I| = \begin{vmatrix}2-\lambda & 0\\ 1 & 2-\lambda\end{vmatrix} = (2-\lambda)^2 = 0$

$\therefore \lambda = 2$

$\therefore$ eign vectors for $\lambda=2$: $\begin{bmatrix}0&0\\1&0\end{bmatrix}$ $\therefore \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\x_2\end{bmatrix} = x_2\begin{bmatrix}0\\1\end{bmatrix}$

Since we have one eign vector for $A\ (2\times2)$ $\therefore$ A is not diagonalizable

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$\frac{10}{284}$ $P(\lambda) = |A-\lambda I| = \begin{vmatrix}-1-\lambda & 4 & -2\\ -3 & 4-\lambda & 0\\ -3 & 1 & 3-\lambda\end{vmatrix} = -(1+\lambda)(4-\lambda)(3-\lambda)+0+6-6(4-\lambda)+12(3-\lambda)-0$

$= -(\lambda+1)(\lambda-4)(\lambda-3)+6-24+6\lambda+36-12\lambda =$

$= -(\lambda+1)(\lambda-4)(\lambda-3)+18-6\lambda = -(\lambda-3)[\lambda^2-3\lambda-4+6]$

$= -(\lambda-3)(\lambda^2-3\lambda+2) = -(\lambda-3)(\lambda-2)(\lambda-1) = 0$

$\therefore \lambda \in \{1, 2, 3\}$

$\because$ A is 3×3 matrix with three distinct λ's $\therefore$ A is diagonalizable

eign vector for $\lambda=1$: $\begin{bmatrix}-2&4&-2\\-3&3&0\\-3&1&2\end{bmatrix} \xrightarrow[\substack{3R_1-2R_2\\ R_2-R_3}]{R_1/(-2)} \begin{bmatrix}1&-2&1\\0&6&-6\\0&2&-2\end{bmatrix} \xrightarrow[R_2-3R_3]{R_2/6}$

$\begin{bmatrix}1&-2&1\\0&1&-1\\0&0&0\end{bmatrix}$ $\therefore x_2-x_3=0$ $\therefore x_2=x_3$ & $x_1-2x_2+x_3=0$ $\therefore x_1=2x_2-x_3=2x_3-x_3=x_3$

$\therefore P_1=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_3\\x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$

eign vector for $\lambda=2$: $\begin{bmatrix}-3&4&-2\\-3&2&0\\-3&1&1\end{bmatrix} \xrightarrow[R_2-R_3]{R_1-R_2} \begin{bmatrix}-3&4&-2\\0&2&-2\\0&1&-1\end{bmatrix} \xrightarrow[R_2-2R_3]{R_2/2} \begin{bmatrix}-3&4&-2\\0&1&-1\\0&0&0\end{bmatrix}$

$\therefore x_2-x_3=0$ $\therefore x_2=x_3$ & $-3x_1+4x_2-2x_3=0$ $\therefore 3x_1=4x_3-2x_3=2x_3$

$\therefore P_2=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2x_3/3\\x_3\\x_3\end{bmatrix}=\frac{x_3}{3}\cdot\begin{bmatrix}2\\3\\3\end{bmatrix}$

eign vector for $\lambda=3$: $\begin{bmatrix}-4&4&-2\\-3&1&0\\-3&1&0\end{bmatrix} \xrightarrow{R_1/2} \xrightarrow{} \xrightarrow{R_2-R_3} \begin{bmatrix}-2&2&-1\\-3&1&0\\0&0&0\end{bmatrix}$ $\therefore -3x_1+x_2=0$ $\therefore x_2=3x_1$

& $-2x_1+2x_2-x_3=0$ $\therefore -2x_1+2(3x_1)-x_3=0$ $\therefore x_3=4x_1$

$\therefore P_3=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\3x_1\\4x_1\end{bmatrix}=x_1\begin{bmatrix}1\\3\\4\end{bmatrix}$

$\therefore P=\begin{bmatrix}1&2&1\\1&3&3\\1&3&4\end{bmatrix}$ & $P^{-1}AP = D = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$.

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$\frac{12}{284}$ $|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = \lambda^2(1-\lambda) = 0 \quad \therefore \lambda = 0 \text{ or } 1$

eign-vectors for $\lambda = 0$: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \Rightarrow -3x_1 = x_3 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

eign vectors for $\lambda = 1$: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \therefore x_1 = x_2 = 0 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ $\therefore$ A is diagonalizable and $P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\frac{14}{284}$ A would be diagonalizable iff it has 4 Linearly Indep. eign vectors

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 0 & 0 & 0 \\ 0 & -2-\lambda & 5 & -5 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} = (-2-\lambda)^2(3-\lambda)^2 = 0 \therefore \lambda = 3 \text{ or } -2$$

eign-vectors for $\lambda = 3$: $\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\therefore -5x_1 = 0 \quad \therefore x_1 = 0$ & $-5x_2 + 5x_3 - 5x_4 = 0$

$\therefore x_2 = x_3 - x_4$ Let $x_4 = t$, $x_3 = s$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ s-t \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $\therefore$ two linearly indep. eign vectors.

eign-vectors for $\lambda = -2$: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \Rightarrow x_4 = 0, x_3 = 0, x_1 = t, x_2 = s$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\therefore$ two linearly indep. eign vectors.

$\therefore$ A is diagonalizable & $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ & $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

$\frac{18}{284}$ (a) $(P^{-1}AP)^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}AIAP = P^{-1}A^2P$

(b) $(P^{-1}AP)^3 = (P^{-1}AP)^2(P^{-1}AP) = P^{-1}A^2PP^{-1}AP = P^{-1}A^2IAP = P^{-1}A^3P$

$\therefore$ Similarly we can conclude that $(P^{-1}AP)^k = P^{-1}A^kP$, k integer

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$\boxed{\frac{19}{285}}$ $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) = 0 \quad \therefore \lambda = 1 \text{ or } 2$

eign vectors for $\lambda=1$: $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow x_1 = x_2 \quad \therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

" " " $\lambda=2$: $\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \Rightarrow x_1 = 0 \quad \therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \therefore P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = D$

$\therefore$ Using $\boxed{\frac{18b}{284}}$: $P^{-1}A^{10}P = (P^{-1}AP)^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{10} = \begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} = D^{10}$

$\therefore A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \frac{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 1 & 0 \\ 1 & 1024 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1023 & 1024 \end{bmatrix}$

$\boxed{\frac{1a}{290}}$ $\because$ Symmetric $\therefore$ Orthogonally diagonalizable

$P(\lambda) = |A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda(\lambda-2) = 0 \quad \therefore \lambda = 0 \text{ or } 2$

eign space $\lambda = 0$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow[R_1 - R_2]{} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \therefore x_1 + x_2 = 0 \quad \therefore x_1 = -x_2 \quad \therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = -x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

eign space $\lambda = 2$: $\begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \therefore x_1 - x_2 = 0 \quad \therefore x_1 = x_2 \quad \therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ & $P^{-1}AP = D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

$\boxed{\frac{16}{290}}$

$$P(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix} = (1-\lambda)^2(-2-\lambda) + 16 + 16 - 4(1-\lambda) + 16(2+\lambda) - 4(1-\lambda) =$$

$$= -(1-\lambda)^2(2+\lambda) + 32 - 8 + 8\lambda + 32 + 16\lambda =$$

$$= -(1-\lambda)^2(2+\lambda) + 56 + 24\lambda = -(2+\lambda)(1-2\lambda+\lambda^2) + 24\lambda + 56$$

$$= -\lambda^3 + 3\lambda - 2 + 24\lambda + 56 = -\lambda^3 + 27\lambda + 54 = -(\lambda+3)^2 \cdot (\lambda-6) = 0$$

$\therefore$ Eign values are $\lambda = -3, 6$

Eign space for $\lambda = -3$: $\begin{bmatrix} 1+3 & -4 & 2 \\ -4 & 1+3 & -2 \\ 2 & -2 & -2+3 \end{bmatrix} \xrightarrow[\frac{R_1}{2} - R_3]{\frac{R_1}{2},\ R_1+R_2} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore\ 2x_1 - 2x_2 + x_3 = 0 \quad \therefore\ x_3 = -2x_1 + 2x_2$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_1 + 2x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\because$ A is symmetric $\therefore$ orthogonally diagonalizable.

$\therefore\ P_1 = \dfrac{\langle 1, 0, -2 \rangle}{\sqrt{1^2 + 0^2 + 2^2}} = \dfrac{1}{\sqrt{5}} \langle 1, 0, -2 \rangle$

& $P_2 = \dfrac{\langle 0,1,2 \rangle \cdot \frac{1}{\sqrt{5}} \langle 1,0,-2 \rangle * \frac{1}{\sqrt{5}} \langle 1,0,-2 \rangle - \langle 0,1,2 \rangle}{\| \quad " \quad \|} =$

$= \dfrac{-\frac{4}{5} \langle 1,0,-2 \rangle - \frac{5}{5} \langle 0,1,2 \rangle}{\| \quad " \quad \|} = \dfrac{\langle -4, -5, -2 \rangle}{-\sqrt{4^2 + 5^2 + 2^2}} = \dfrac{\langle +4, +5, +2 \rangle}{\sqrt{45}}$

$\therefore$ Basis for eign space for $\lambda = -3$ is $\frac{1}{\sqrt{5}} \langle 1, 0, -2 \rangle$ & $\frac{1}{3\sqrt{5}} \langle 4, 5, 2 \rangle$

Eign space for $\lambda = 6$: $\begin{bmatrix} 1-6 & -4 & 2 \\ -4 & 1-6 & -2 \\ 2 & -2 & -2-6 \end{bmatrix} \xrightarrow[R_2 + 2R_3]{-R_1,\ -4R_1 + 5R_2} \begin{bmatrix} 5 & 4 & -2 \\ 0 & -9 & -18 \\ 0 & -9 & -18 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_2/(-9)} \begin{bmatrix} 5 & 4 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore\ x_2 + 2x_3 = 0 \quad \therefore\ x_2 = -2x_3$ & $5x_1 + 4(-2x_3) - 2x_3 = 0 \quad \therefore\ x_1 = 2x_3$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $\therefore$ P_3 = basis for eign space for $\lambda = 6 = \frac{1}{3} \langle 2, -2, 1 \rangle$ $\therefore\ P = \begin{bmatrix} 1/\sqrt{5} & 4/3\sqrt{5} & 2/3 \\ 0 & 5/3\sqrt{5} & -2/3 \\ -2/\sqrt{5} & 2/3\sqrt{5} & 1/3 \end{bmatrix}$ & $P^{-1}AP = D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$\frac{1d}{290}$ $|A-\lambda I| = \begin{vmatrix} \frac{10}{3}-\lambda & -4/3 & 0 & -4/3 \\ -4/3 & -\frac{5}{3}-\lambda & 0 & 1/3 \\ 0 & 0 & -2-\lambda & 0 \\ -4/3 & 1/3 & 0 & -\frac{5}{3}-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} \frac{10}{3}-\lambda & -4/3 & -4/3 \\ -4/3 & -\frac{5}{3}-\lambda & 1/3 \\ -4/3 & 1/3 & -\frac{5}{3}-\lambda \end{vmatrix} =$

$= -(2+\lambda)\left[(\frac{10}{3}-\lambda)(-\frac{5}{3}-\lambda)^2 + \frac{4}{3}\cdot\frac{1}{3}\cdot\frac{4}{3} * 2 - \frac{4}{3}\cdot\frac{4}{3}(-\frac{5}{3}-\lambda) - \frac{4}{3}\cdot\frac{4}{3}\cdot(-\frac{5}{3}-\lambda) - \frac{(\frac{10}{3}-\lambda)}{9}\right]$

$= \frac{-(2+\lambda)}{27}\left[(10-3\lambda)(5+3\lambda)^2 + 32 + 32(5+3\lambda) - (10-3\lambda)\right] =$

$= \frac{-(2+\lambda)}{27}\cdot\left[(10-3\lambda)(25+30\lambda+9\lambda^2) + 182 + 99\lambda\right] =$

$= \frac{1}{27}\cdot(2+\lambda)\cdot(432+324\lambda-27\lambda^3) = -\frac{1}{27}\cdot(2+\lambda)^2(216+54\lambda-27\lambda^2)$

$= -\frac{1}{27}(2+\lambda)^2 (27)(8+2\lambda-\lambda^2) = (\lambda+2)^2(\lambda^2-2\lambda-8) = (\lambda+2)^3(\lambda-4) = 0$

$\therefore$ By theorem 7b $\therefore$ The dimension of eign space of $\lambda=-2$ is 3

& The dimension of eignspace of $\lambda = 4$ is one.

$\frac{4}{291}$ $P(\lambda) = |A-\lambda I| = \begin{vmatrix} -7-\lambda & 24 \\ 24 & 7-\lambda \end{vmatrix} = -(7+\lambda)(7-\lambda) - 24^2 = (\lambda+7)(\lambda-7) - 24^2 =$

$= \lambda^2 - 7^2 - 24^2 = \lambda^2 - 49 - 576 = \lambda^2 - 625 = (\lambda-25)(\lambda+25) = 0$

$\therefore \lambda \in \{25, -25\}$

$\therefore$ eign vector for $\lambda = 25$: $\begin{bmatrix} -32 & 24 \\ 24 & -18 \end{bmatrix} \xrightarrow[R_2/(6)]{R_1/(-8)} \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} \xRightarrow[R_1-R_2]{} \begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$

$\therefore 4x_1 - 3x_2 = 0$

$\therefore 4x_1 = 3x_2 \quad \therefore x_1 = 3x_2/4$

$\therefore P_{25} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2/4 \\ x_2 \end{bmatrix} = \frac{x_2}{4}\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{5x_2}{4}\cdot\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}.$

& eign vector for $\lambda = -25$: $\begin{bmatrix} 18 & 24 \\ 24 & 32 \end{bmatrix} \xrightarrow[R_2/8]{R_1/6} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \xRightarrow[R_1-R_2]{} \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

$\therefore 3x_1 + 4x_2 = 0 \quad \therefore x_1 = -4x_2/3$

$\therefore P_{-25} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_2/3 \\ x_2 \end{bmatrix} = -x_2/3\cdot\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \frac{-5x_2}{3}\cdot\begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}.$

$\therefore P = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}/5$ & $P^{-1}AP = D = \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix}$

OR: $P = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}$ & $P^{-1}AP = D = \begin{bmatrix} -25 & 0 \\ 0 & 25 \end{bmatrix}$

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$\frac{5}{291}$

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 0 & -36 \\ 0 & -3-\lambda & 0 \\ -36 & 0 & -23-\lambda \end{vmatrix} = (-3-\lambda)\begin{vmatrix} -2-\lambda & -36 \\ -36 & -23-\lambda \end{vmatrix} = (-3-\lambda)\left[(-2-\lambda)(-23-\lambda) - 36^2\right]$$

$$= -(\lambda+3)(\lambda^2+25\lambda+46-1296) = -(\lambda+3)(\lambda^2+25\lambda-1250) =$$

$$= -(\lambda+3)(\lambda-25)(\lambda+50) = 0 \quad \therefore \lambda = -3, 25, -50$$

eigen-vectors for $\lambda=-3$: $\begin{bmatrix} 1 & 0 & -36 \\ 0 & 0 & 0 \\ -36 & 0 & -20 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -36 \\ 0 & 0 & 0 \\ 0 & 0 & -36^2-20 \end{bmatrix} \therefore x_3 = x_1 = 0 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

eigen-vectors for $\lambda=25$: $\begin{bmatrix} -27 & 0 & -36 \\ 0 & -28 & 0 \\ -36 & 0 & -48 \end{bmatrix} \overset{R_1/9,\ R_3/12}{\Rightarrow} \begin{bmatrix} -3 & 0 & -4 \\ 0 & -28 & 0 \\ -3 & 0 & -4 \end{bmatrix} \therefore 3x_1 = -4x_3,\ x_2 = 0 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \frac{x_3}{+3}\begin{bmatrix} -4 \\ 0 \\ +3 \end{bmatrix}$

eigen-vectors for $\lambda=-50$: $\begin{bmatrix} 48 & 0 & -36 \\ 0 & 47 & 0 \\ -36 & 0 & 27 \end{bmatrix} \overset{R_1/12,\ R_3/(-9)}{\Longrightarrow} \begin{bmatrix} 4 & 0 & -3 \\ 0 & 47 & 0 \\ 4 & 0 & -3 \end{bmatrix} \therefore 4x_1 = 3x_3,\ x_2 = 0 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3x_3}{4} \\ 0 \\ x_3 \end{bmatrix} = \frac{x_3}{4}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 0 & -\frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \\ 0 & +\frac{3}{5} & \frac{4}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & -4 & 3 \\ 5 & 0 & 0 \\ 0 & 3 & 4 \end{bmatrix}$ ($\det P = 1$) orthogonally diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{bmatrix}$

$\frac{6}{291}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore |A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(1-\lambda)^2 + \lambda =$$

$$= \lambda(1-(1-\lambda)^2) =$$

$$= \lambda(1-(1-\lambda))(1+(1-\lambda))$$

$$= \lambda^2(2-\lambda)$$

$\therefore \lambda = 0, 2$

eigen value of $\lambda = 0$:

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \therefore x_1 = -x_2 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{P_1} + x_3\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{P_2} \quad P_1 \cdot P_2 = 0 \therefore P_1 \perp P_2$

eigen value of $\lambda = 2$:

$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \therefore x_3 = 0,\ x_1 = x_2 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\therefore P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$ & $\det P = 1$

$\therefore P$ orthogonally diagonalizes A & $P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$\frac{9}{291}$

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -2 & 0 & 0 \\ -2 & 2-\lambda & 0 & 0 \\ 0 & 0 & 5-\lambda & -2 \\ 0 & 0 & -2 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda)\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} - (-2)(-2)\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} =$$

$$= [(5-\lambda)(2-\lambda)-4]\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = [(5-\lambda)(2-\lambda)-4]^2 = (10-7\lambda+\lambda^2-4)^2$$

$$= (6-7\lambda+\lambda^2)^2 = [(6-\lambda)(1-\lambda)]^2 = (\lambda-6)^2(\lambda-1)^2 = 0 \therefore \lambda = 1 \text{ or } 6$$

eign-vectors for $\lambda = 1$:

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{matrix} R_1/2 \\ R_1+2R_2 \\ R_3/2 \\ R_3+2R_4 \end{matrix} \Rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \therefore 2x_1 = x_2,\ 2x_3 = x_4 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{P_1} + x_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}}_{P_2}$$

eign vectors for $\lambda = 6$:

$$\begin{bmatrix} -1 & -2 & 0 & 0 \\ -2 & -4 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix} \begin{matrix} -R_1 \\ R_2-2R_1 \\ -R_3 \\ R_4-2R_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \therefore x_1 = -2x_2,\ x_3 = -2x_4 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{P_3} + x_4 \underbrace{\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{P_4}$$

$P_1 \cdot P_2 = 0$, $P_3 \cdot P_4 = 0$ $\therefore$ P_1, P_2, P_3 & P_4 are orthogonal (because A is symmetric see theorem 6)

$\therefore \left[\frac{P_1}{\sqrt5} \middle| \frac{P_2}{\sqrt5} \middle| \frac{P_3}{\sqrt5} \middle| \frac{P_4}{\sqrt5}\right]$ is orthonormal

$$\therefore P = \frac{1}{\sqrt5}\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 2 & 0 & 1 \end{bmatrix} \text{ orthogonally diagonalizes } A.$$

$$\& \ P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$\frac{10}{291}$

Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, $b \neq 0$ $\therefore |A-\lambda I| = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 =$

$= (a-\lambda-b)(a-\lambda+b) = 0 \quad \therefore \lambda = a-b \text{ or } \lambda = a+b$

eign vectors for $\lambda = a-b$: $\begin{bmatrix} a-a+b & b \\ b & a-a+b \end{bmatrix} \therefore x_1 = -x_2 \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{P_1}$

" " " $\lambda = a+b$: $\begin{bmatrix} a-a-b & b \\ b & a-a-b \end{bmatrix} \therefore x_1 = x_2 \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{P_2}$

$\therefore \left[\frac{P_1}{\sqrt2} \middle| \frac{P_2}{\sqrt2}\right]$ is orthonormal

$$\therefore P = \frac{1}{\sqrt2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ orthogonally diagonalizes } A.$$

$\boxed{\frac{7}{292}}$ (a) $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ $\therefore |A - \lambda I| = \begin{vmatrix} 3-\lambda & 6 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 6 = 6 - 5\lambda + \lambda^2 - 6$
$= \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$

$\therefore A(A - 5I) = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}\left(\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore$ OK

(b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

$\therefore |A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = \lambda^2(3-\lambda) + 1 + \lambda(-3) = 3\lambda^2 - \lambda^3 + 1 - 3\lambda =$
$= (-\lambda^3 + 1) - 3\lambda(1-\lambda) = (-\lambda+1)(\lambda^2+\lambda+1) - 3\lambda(-\lambda+1)$
$= (1-\lambda)\left[\lambda^2 + \lambda + 1 - 3\lambda\right] = (1-\lambda)(\lambda^2 - 2\lambda + 1) = (1-\lambda)(\lambda-1)^2 = -(\lambda-1)^3 = 0$

$\therefore -(A - I)^3 = -\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix}^3 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore$ OK.

$\boxed{\frac{9}{292}}$ from (a) above, $\therefore \lambda^2 - 5\lambda = 0$ $\therefore \lambda^2 = 5\lambda$ $\therefore A^2 = 5A = \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$.
$\therefore A^3 = 5A^2 = 5 * 5A = 25A = \begin{bmatrix} 75 & 150 \\ 25 & 50 \end{bmatrix}$.
$\therefore A^4 = 5A^3 = 5 * 25A = 125A = \begin{bmatrix} 375 & 750 \\ 125 & 250 \end{bmatrix}$.
& $A^5 = 5A^4 = 5 * 125A = 625A = \begin{bmatrix} 1875 & 3750 \\ 625 & 1250 \end{bmatrix}$.

$\frac{1}{300}$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ $\therefore |A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} =$

$= (1-\lambda)(3-\lambda) - 8 = 3 - 4\lambda + \lambda^2 - 8 = \lambda^2 - 4\lambda - 5 =$

$= (\lambda - 5)(\lambda + 1) = 0 \quad \therefore \lambda = 5$ or -1

eign-vectors for $\lambda = 5$: $\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \Rightarrow \therefore x_1 = x_2 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eign vectors for $\lambda = -1$: $\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \Rightarrow \therefore x_1 = -2x_2 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. Let $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ or $Y = PU$

$\therefore Y' = PU'$ but $Y' = AY \quad \therefore PU' = APU$

$\therefore U' = P^{-1}APU = DU = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \quad \therefore \begin{cases} 5u_1 = u_1' \\ -1u_2 = u_2' \end{cases} \quad \therefore u_1 = c_1 e^{5x}$ & $u_2 = c_2 e^{-x}$

$\therefore U = \begin{bmatrix} c_1 e^{5x} \\ c_2 e^{-x} \end{bmatrix} \quad \therefore Y = PU = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{5x} \\ c_2 e^{-x} \end{bmatrix} = \begin{bmatrix} c_1 e^{5x} - 2c_2 e^{-x} \\ c_1 e^{5x} + c_2 e^{-x} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

(a) $\therefore y_1 = c_1 e^{5x} - 2c_2 e^{-x}$ & $y_2 = c_1 e^{5x} + c_2 e^{-x}$

(b) $Y(0) = 0 = PU(0) ; \therefore P\,U(0) = 0 \quad \therefore U(0) = 0 \quad \therefore \begin{bmatrix} c_1 e^{5(0)} \\ c_2 e^{-(0)} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \therefore c_1 = c_2 = 0$

$\therefore y_1 = 0$ & $y_2 = 0$ is the solution for $y_1(0) = y_2(0) = 0$

$\frac{2}{300}$ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = Y' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} Y = AY$

$\therefore P(\lambda) = |A-\lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 12 = 5 - 6\lambda + \lambda^2 - 12$

$= \lambda^2 - 6\lambda - 7 = (\lambda-7)(\lambda+1) = 0 \quad \therefore \lambda = 7, -1$

Eign space for $\lambda = 7$:

$\therefore \begin{bmatrix} 1-7 & 3 \\ 4 & 5-7 \end{bmatrix} \xrightarrow[R_2+\frac{2}{3}R_1]{\frac{R_1}{-3}} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \therefore 2x_1 - x_2 = 0 \therefore x_2 = 2x_1 \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore$ basis for eign space for $\lambda = 7$ is $\langle 1, 2 \rangle$

Eign space for $\lambda = -1$

$\therefore \begin{bmatrix} 1+1 & 3 \\ 4 & 5+1 \end{bmatrix} \xrightarrow[2R_1 - R_2]{} \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \therefore 2x_1 + 3x_2 = 0 \therefore x_2 = \frac{-2x_1}{3} \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{-2x_1}{3} \end{bmatrix} = \frac{x_1}{3} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\therefore$ basis for eign space for $\lambda = -1$ is $\langle 3, -2 \rangle$

$\therefore P = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$ & $D = P^{-1}AP = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$

Let $Y = PU \quad \therefore Y' = PU' \quad \therefore PU' = APU$

$\therefore U' = P^{-1}AP\, U = DU \quad \therefore \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 7u_1 \\ -u_2 \end{bmatrix}$

$\therefore u_1' = 7u_1 \Rightarrow u_1 = Ae^{7t}$ & $u_2' = -u_2 \Rightarrow u_2 = Be^{-t}$

$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = PU = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} Ae^{7t} \\ Be^{-t} \end{bmatrix} = \begin{bmatrix} Ae^{7t} + 3Be^{-t} \\ 2Ae^{7t} - 2Be^{-t} \end{bmatrix}$

$y_1(0) = 2 \quad \therefore A + 3B = 2$ & $y_2'(0) = 1 \quad \therefore 14A + 2B = 1$

Solving for A & B $\quad \therefore (2-42)B = 1 - 28 \quad \therefore B = +27/40 \quad \therefore A = -1/40$

$\therefore y_1 = (81e^{-t} - e^{7t})/40$ & $y_2 = -(27e^{-t} + e^{7t})/20$

$\boxed{\frac{3}{301}}$

(a) $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ or $Y' = AY$; $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)^2 + 2(1-\lambda)$

$= (1-\lambda)(4 - 5\lambda + \lambda^2 + 2) = (1-\lambda)(6 - 5\lambda + \lambda^2)$

$= (1-\lambda)(3-\lambda)(2-\lambda) = 0 \quad \therefore \lambda = 1, 2, 3$

eigen-vectors for $\lambda = 1$: $\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ $\therefore x_1 = 0 \therefore x_3 = 0 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

" " " $\lambda = 2$: $\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore x_2 = x_3,\ 2x_1 = -x_3 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{x_3}{2} \\ x_3 \\ x_3 \end{bmatrix} = -\frac{x_3}{2} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

" " " $\lambda = 3$ $\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2/(-2)} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore x_1 = -x_3,\ x_2 = -x_1 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix}$ Let $Y = PU$ $\therefore Y' = PU' = AY = APU \quad \therefore U' = P^{-1}APU = DU$

but $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\therefore \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ or $\begin{matrix} u_1' = u_1 \\ u_2' = 2u_2 \\ u_3' = 3u_3 \end{matrix}$ $\therefore$ $\begin{matrix} u_1 = c_1 e^t \\ u_2 = c_2 e^{2t} \\ u_3 = c_3 e^{3t} \end{matrix}$

$\therefore U = \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \\ c_3 e^{3t} \end{bmatrix}$ $\therefore Y = PU = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \\ c_3 e^{3t} \end{bmatrix} = \begin{bmatrix} c_2 e^{2t} + c_3 e^{3t} \\ c_1 e^t - 2c_2 e^{2t} - c_3 e^{3t} \\ -2c_2 e^{2t} - c_3 e^{3t} \end{bmatrix}$

$\therefore y_1 = c_2 e^{2t} + c_3 e^{3t}$, $y_2 = c_1 e^t - 2c_2 e^{2t} - c_3 e^{3t}$ & $y_3 = -2c_2 e^{2t} - c_3 e^{3t}$.

(b) $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{t=0} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = PU_{t=0} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 0 & -2 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ R_3+2R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ $\therefore$ $\begin{matrix} c_3 = -2 \\ c_2 = 2 - 1 = 1 \\ c_1 = 2 + (-2) + 1 \\ = 1 \end{matrix}$

$\therefore y_1 = e^{2t} - 2e^{3t}$, $y_2 = e^t - 2e^{2t} + 2e^{3t}$ & $y_3 = -2e^{2t} + 2e^{3t}$

$\frac{6}{301}$ Let $y_1 = y$, $y_2 = y'$ & $y_3 = y''$ $\therefore \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ 6y'' - 11y' + 6y \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ 6y_3 - 11y_2 + 6y_1 \end{bmatrix}$

(from D.Eq.)

$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \therefore Y' = AY; \therefore |A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & -11 & 6-\lambda \end{vmatrix} = \lambda^2(6-\lambda) + 6 - 11\lambda =$

$= 6\lambda^2 - \lambda^3 - 11\lambda + 6 = (\lambda - 1)(-\lambda^2 + 5\lambda - 6) =$

$= -(\lambda - 1)(\lambda - 3)(\lambda - 2) = 0 \quad \therefore \lambda = 1, 2, 3$

eign vector for $\lambda = 1$: $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \xrightarrow[R_3 + 6R_1]{} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -5 & 5 \end{bmatrix} \therefore x_2 = x_3$ & $x_1 = x_2 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

" " " " $\lambda = 2$: $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \xrightarrow[R_3 + 3R_1]{} \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -8 & 4 \end{bmatrix} \therefore 2x_2 = x_3$ & $2x_1 = x_2 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2/2 \\ x_2 \\ 2x_2 \end{bmatrix} = \frac{x_2}{2} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

" " " $\lambda = 3$: $\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix} \xrightarrow[R_3 + 2R_1]{} \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & -9 & 3 \end{bmatrix} \therefore 3x_1 = x_2$ & $3x_2 = x_3 \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2/3 \\ x_2 \\ 3x_2 \end{bmatrix} = \frac{x_2}{3} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$. Let $Y = PU \therefore Y' = PU' = AY = APU \therefore U' = P^{-1}APU$
$= DU$

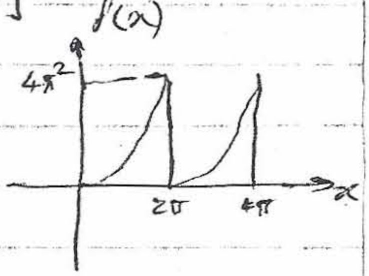
but $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \therefore \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow \begin{matrix} u_1' = u_1 \\ u_2' = 2u_2 \\ u_3' = 3u_3 \end{matrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \\ c_3 e^{3t} \end{bmatrix}$

$\therefore U = \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \\ c_3 e^{3t} \end{bmatrix} \therefore Y = PU = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \\ c_3 e^{3t} \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ c_1 e^t + 2c_2 e^{2t} + 3c_3 e^{3t} \\ c_1 e^t + 4c_2 e^{2t} + 9c_3 e^{3t} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$

$\therefore y = c_1 e^t + c_2 e^{2t} + c_3 e^{3t}$ is the solution for $y''' - 6y'' + 11y' - 6y = 0$

$\boxed{\frac{2}{307}}$

$f(x) = x^2$ periodic over $x \in [0, 2\pi]$



$$av. = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{x^3}{2\pi * 3}\Big|_0^{2\pi} = \frac{(2\pi)^3}{3(2\pi)} = \frac{4\pi^2}{3}$$

sin coeff. $a_n = \frac{2}{2\pi}\int_0^{2\pi} x^2 \sin nx\, dx =$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\, d\frac{\cos nx}{-n} = \frac{1}{\pi}\left[\frac{x^2\cos nx}{-n}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\cos nx}{-n}\cdot 2x\,dx\right] =$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2}{-n} + \frac{2}{n}\int_0^{2\pi} x\, d\frac{\sin nx}{n}\right] = \frac{2}{n\pi}\left[-2\pi^2 + x\frac{\sin nx}{n}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{2}{n\pi}\left[-2\pi^2 + 0 + \frac{\cos nx}{n2}\Big|_0^{2\pi}\right] = \frac{2}{n\pi}\left[-2\pi^2 + \frac{1-1}{n^2}\right] = \frac{-4\pi}{n}$$

cos coeff. $b_n = \frac{2}{2\pi}\int_0^{2\pi} x^2\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\, d\frac{\sin nx}{n} =$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\sin nx}{n}\cdot 2x\,dx\right] = \frac{1}{\pi}\left[0 - 0 - \frac{2}{n}\int_0^{2\pi} x\, d\frac{\cos nx}{-n}\right] =$$

$$= \frac{2}{\pi n^2}\cdot\left[x\cos nx\Big|_0^{2\pi} - \int_0^{2\pi}\cos nx\,dx\right] = \frac{2}{\pi n^2}\cdot\left[2\pi - 0 - \frac{\sin nx}{n}\Big|_0^{2\pi}\right] =$$

$$= \frac{2}{\pi n^2}\cdot\left(2\pi - \frac{0-0}{n}\right) = \frac{4}{n^2}$$

$$\therefore\ x^2 \text{ over } [0, 2\pi] = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} a_n\sin nx + b_n\cos nx =$$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} -\frac{4\pi}{n}\sin nx + \frac{4}{n^2}\cos nx$$

$$\begin{aligned} = \frac{4\pi^2}{3} &- 4\pi\sin x + 4\cos x \\ &- 2\pi\sin 2x + \cos 2x \\ &- \frac{4\pi}{3}\sin 3x + \frac{4}{9}\cos 3x \\ &- \pi\sin 4x + \frac{1}{4}\cos 4x + \cdots \end{aligned}$$

3
307

$x \approx a + b e^x \qquad x \in [0, 1]$

a) $\langle x, 1 \rangle = \int_0^1 x \cdot 1 \cdot dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$

$\langle 1, 1 \rangle = \int_0^1 1 \cdot 1 \cdot dx = x \Big|_0^1 = 1$

$\langle e^x, 1 \rangle = \int_0^1 e^x \cdot 1 dx = e^x \Big|_0^1 = e - 1$

$\langle x, e^x \rangle = \int_0^1 x e^x dx = \int_0^1 x \, de^x = x e^x \Big|_0^1 - \int_0^1 e^x dx = e - 0 - e^x \Big|_0^1 = \underbrace{e - e + 1}_{=1}$

$\langle 1, e^x \rangle = \langle e^x, 1 \rangle = e - 1$

$\langle e^x, e^x \rangle = \int_0^1 e^x \cdot e^x dx = \frac{e^{2x}}{2} \Big|_0^1 = \frac{e^2 - 1}{2}$

$\therefore \langle x, 1 \rangle \approx a \langle 1, 1 \rangle + b \langle e^x, 1 \rangle \Rightarrow \frac{1}{2} \approx a \cdot 1 + b \cdot (e-1)$

& $\langle x, e^x \rangle \approx a \langle 1, e^x \rangle + b \langle e^x, e^x \rangle \Rightarrow 1 \approx a \cdot (e-1) + b \left(\frac{e^2-1}{2} \right)$

$\therefore \Delta = \frac{e^2-1}{2} - (e-1)^2 = \frac{e^2 - 1 - 2e^2 + 4e - 2}{2} = \frac{-e^2 + 4e - 3}{2} = -\frac{1}{2}(e-3)(e-1)$

& $\Delta_a = \frac{e^2-1}{4} - (e-1) = \frac{e^2 - 1 - 4e + 4}{4} = \frac{e^2 - 4e + 3}{4} = \frac{1}{4}(e-3)(e-1)$

& $\Delta_b = 1 - \frac{1}{2}(e-1) = \frac{2 - e + 1}{2} = \frac{3 - e}{2} = -\frac{1}{2}(e-3)$

$\therefore a = \frac{\Delta a}{\Delta} = \frac{(e-3)(e-1)/4}{-(e-3)(e-1)/2} = -\frac{2}{4} = -1/2$

& $b = \frac{\Delta b}{\Delta} = \frac{-(e-3)/2}{-(e-3)(e-1)/2} = \frac{1}{e-1}$

$\therefore x \approx -\frac{1}{2} + \frac{e^x}{e-1}$

b) Mean Square Error $= \int_0^1 \left[x - \left(-\frac{1}{2} + \frac{e^x}{e-1} \right) \right]^2 dx = \int_0^1 \left(x^2 + \frac{1}{4} + \frac{e^{2x}}{(e-1)^2} + x - \frac{2xe^x}{e-1} - \frac{e^x}{e-1} \right) dx$

$= \frac{x^3}{3} + \frac{x}{4} + \frac{e^{2x}}{2(e-1)^2} + \frac{x^2}{2} - \frac{2}{e-1}\left(x e^x - e^x \right) - \frac{e^x}{e-1} \Big|_0^1 = \frac{1}{3} + \frac{1}{4} + \frac{e^2 - 1}{2(e-1)^2} + \frac{1}{2} - \frac{2 + e - 1}{e-1}$

$= \frac{13(e-1) + 6(e+1) - 12(e+1)}{12(e-1)} = \frac{7e - 19}{12(e-1)} = 1.357 * 10^{-3}$

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$\boxed{\frac{4}{307}}$ ⓐ $e^x \simeq a_0 + a_1 x$ Let $w = e^x$, $u_1 = 1$, $u_2 = x$

$\therefore w \simeq a_0 u_1 + a_1 u_2$ $\therefore \langle w, u_1\rangle = a_0\langle u_1, u_1\rangle + a_1\langle u_2, u_1\rangle$ ①

& $\langle w, u_2\rangle = a_0\langle u_1, u_2\rangle + a_1\langle u_2, u_2\rangle$ ②

$\therefore \langle w, u_1\rangle = \int_0^1 w \cdot u_1 \, dx = \int_0^1 e^x \cdot 1 \cdot dx = e^x\Big|_0^1 = e-1$

& $\langle w, u_2\rangle = \int_0^1 w \cdot u_2 \, dx = \int_0^1 e^x \cdot x \, dx = xe^x - e^x\Big|_0^1 = e - (e-1) = 1$

& $\langle u_1, u_1\rangle = \int_0^1 u_1^2 \, dx = \int_0^1 1^2 dx = x\Big|_0^1 = 1$

& $\langle u_1, u_2\rangle = \langle u_2, u_1\rangle = \int_0^1 1 \cdot x \cdot dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

& $\langle u_2, u_2\rangle = \int_0^1 u_2^2 \, dx = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

① becomes $e - 1 = a_0 \cdot 1 + a_1 \cdot \frac{1}{2}$ & ② becomes $1 = a_0 \cdot \frac{1}{2} + a_1 \cdot \frac{1}{3}$

& $\begin{bmatrix} e-1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ $\therefore \Delta = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, $D_{a_0} = \frac{e-1}{3} - \frac{1}{2} = \frac{2e-5}{6}$, $D_{a_1} = 1 - \frac{e-1}{2} = \frac{3-e}{2}$

$\therefore a_0 = \frac{2e-5}{6} * 12 = 4e - 10$, $a_1 = \frac{3-e}{2} \cdot 12 = 6(3-e)$

$\therefore e^x \simeq 4e - 10 + 6(3-e)x$

ⓑ error $= \int_0^1 (a_0 + a_1 x - e^x)^2 dx = \int_0^1 (a_0^2 + a_1^2 x^2 + e^{2x} + 2a_0 a_1 x - 2a_0 e^x - 2a_1 x e^x) dx =$

$= a_0^2 x + a_1^2 \frac{x^3}{3} + \frac{e^{2x}}{2} + a_0 a_1 x^2 - 2a_0 e^x - 2a_1(xe^x - e^x)\Big|_0^1 = a_0^2 + \frac{a_1^2}{3} + \frac{e^2-1}{2} + a_0 a_1$

$-2a_0(e-1) - 2a_1(\cancel{e} - (\cancel{e}-1)) = (4e-10)^2 + \frac{36(3-e)^2}{3} + \frac{e^2-1}{2} + (4e-10)6(3-e)$

$-2(4e-10)(e-1) - 2*6(3-e) = 16e^2 + 100 - 80e + 108 + 12e^2 - 72e + \frac{e^2-1}{2} + 132e$

$-180 - 24e^2 - 8e^2 - 20 + 28e - 36 + 12e = -\frac{7e^2}{2} - 28\frac{1}{2} + 20e = -\frac{1}{2}(7e^2 - 40e + 57)$

$= -\frac{1}{2}(7e - 19)(e - 3) = \frac{(3-e)(7e-19)}{2} = 3.94 * 10^{-3} = .00394$

$\boxed{\frac{5}{307}}$

$\sin \pi x \cong a_0 + a_1 x + a_2 x^2 \qquad x \in [-1, 1]$

Let $u_1 = 1$, $u_2 = x$, $u_3 = x^2$ & construct an orthonormal basis

$\therefore |u_1|^2 = \int_{-1}^{1} 1 \cdot 1\, dx = \int dx = x \Big]_{-1}^{1} = 2 \quad \therefore |u_1| = \sqrt{2} \qquad \therefore v_1 = \frac{u_1}{|u_1|} = \frac{1}{\sqrt{2}}$

$u_2 = c_1 v_1 + c_2 v_2 \qquad \therefore \langle u_2, v_1 \rangle = c_1 \quad \therefore c_1 = \int_{-1}^{1} u_2 v_1\, dx = \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}} \frac{x^2}{2} \Big]_{-1}^{1}$

$= \frac{1}{2\sqrt{2}} [1 - 1] = 0 \qquad \therefore u_2 = 0 v_1 + c_2 v_2 = c_2 v_2 \quad \therefore c_2 v_2 = u_2 = x$

$\therefore v_2 = \frac{x}{|x|} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \frac{x}{\sqrt{\frac{x^3}{3}\Big]_{-1}^{1}}} = \frac{x}{\sqrt{\frac{1+1}{3}}} = \sqrt{\frac{3}{2}} \cdot x$

$u_3 = c_1' v_1 + c_2' v_2 + c_3' v_3$, $\therefore \langle u_3, v_1 \rangle = c_1'$, $\langle u_3, v_2 \rangle = c_2'$

$\therefore c_1' = \int_{-1}^{1} x^2 \cdot \frac{1}{\sqrt{2}}\, dx = \frac{x^3}{3\sqrt{2}} \Big]_{-1}^{1} = \frac{1+1}{3\sqrt{2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$

$c_2' = \int_{-1}^{1} x^2 \cdot \sqrt{\frac{3}{2}} \cdot x\, dx = \sqrt{\frac{3}{2}} \frac{x^4}{4} \Big]_{-1}^{1} = 0$

$\therefore u_3 = \frac{\sqrt{2}}{3} v_1 + 0 \cdot v_2 + c' v_3 \quad \therefore c' v_3 = u_3 - \frac{\sqrt{2}}{3} v_1 = x^2 - \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}$

$\therefore |c' v_3|^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1} (x^4 - \frac{2x^2}{3} + \frac{1}{9}) dx = \frac{x^5}{5} - \frac{2}{9} x^3 + \frac{x}{9} \Big]_{-1}^{1} = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$

$\therefore v_3 = \frac{c' v_3}{|c' v_3|} = \frac{x^2 - \frac{1}{3}}{\sqrt{8/45}} = \sqrt{\frac{45}{8}} \cdot (x^2 - \frac{1}{3}) = \frac{3}{2}\sqrt{\frac{5}{2}} \cdot (x^2 - \frac{1}{3})$

$\therefore \sin \pi x \cong a_0 u_1 + a_1 u_2 + a_2 u_3 = a_0' v_1 + a_1' v_2 + a_2' v_3$

$\therefore \langle \sin \pi x, v_1 \rangle = a_0'$, $\langle \sin \pi x, v_2 \rangle = a_1'$ & $\langle \sin \pi x, v_3 \rangle = a_2'$

$\therefore a_0' = \int_{-1}^{1} \sin \pi x \cdot \frac{1}{\sqrt{2}}\, dx = \frac{-\cos \pi x}{\pi \sqrt{2}} \Big]_{-1}^{1} = -\frac{1}{\sqrt{2}\pi} [\cos \pi - \cos(-\pi)] = 0$

$a_1' = \int_{-1}^{1} \sin \pi x \cdot \sqrt{\frac{3}{2}}\, x\, dx = \sqrt{\frac{3}{2}} \left[\frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} \right]_{-1}^{1} = \sqrt{\frac{3}{2}} \left[\frac{1+1}{\pi} + \frac{0-0}{\pi^2} \right] = \sqrt{\frac{3}{2}} \cdot \frac{2}{\pi} = \frac{\sqrt{6}}{\pi}$

$a_2' = \int_{-1}^{1} \sin \pi x \cdot \frac{3}{2}\sqrt{\frac{5}{2}} \cdot (x^2 - \frac{1}{3})\, dx = \frac{3}{2}\sqrt{\frac{5}{2}} \left[\frac{-x^2 \cos \pi x}{\pi} + \frac{2}{\pi}\left(\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right) + \frac{1}{3} \cdot \frac{\cos \pi x}{\pi} \right]_{-1}^{1} =$

$= \frac{3}{2}\sqrt{\frac{5}{2}} \left[\frac{1-1}{\pi} + \frac{2}{\pi}\left(\frac{0-0}{\pi} + \frac{-1+1}{\pi^2} \right) + \frac{-1+1}{3\pi} \right] = 0$

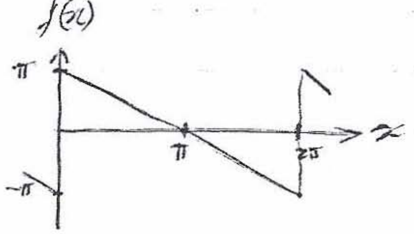
$\therefore$ (a) $\sin \pi x \cong a_0' v_1 + a_1' v_2 + a_2' v_3 = 0 \cdot v_1 + \frac{\sqrt{6}}{\pi} v_2 + 0 v_3 = \frac{\sqrt{6}}{\pi} \cdot \sqrt{\frac{3}{2}}\, x = \sqrt{9}\, \frac{x}{\pi} = \frac{3x}{\pi}$

& (b) mean square error $= \int_{-1}^{1} (\sin \pi x - \frac{3x}{\pi})^2 dx = \int_{-1}^{1} (\sin^2 \pi x - \frac{6x}{\pi} \sin \pi x + \frac{9x^2}{\pi^2}) dx =$

$= \frac{x - \frac{\sin 2\pi x}{2\pi}}{2} \Big|_{-1}^{1} - \frac{6}{\pi} \cdot \frac{2}{\pi} + \frac{9x^3}{3\pi^2} \Big|_{-1}^{1} = \frac{1+1-\frac{0-0}{2\pi}}{2} - \frac{12}{\pi^2} + \frac{3(1+1)}{\pi^2} =$

$= 1 - \frac{12}{\pi^2} + \frac{6}{\pi^2} = 1 - \frac{6}{\pi^2} \simeq 0.3921$

$\frac{8}{307}$ $f(x) = \pi - x$, $x \in [0, 2\pi]$ then repeats.



$\therefore f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$

$\frac{a_0}{2}$ is the average (seen to be) $= 0$

$\therefore a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \cos kx \, dx = 0$ always

because equal areas with opposite signs.

$\therefore b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin kx \, dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \cdot \sin kx \, dx$, because equal areas with same signs

$\therefore b_k = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \sin kx \, dx = \frac{2}{\pi}\left[\frac{-\pi \cos kx}{k} + \frac{x \cos kx}{k} - \frac{\sin kx}{k^2}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\frac{-\pi(-1-1)}{k} + \frac{\pi(-1) - 0(1)}{k} - \frac{0-0}{k^2}\right] = \frac{2}{\pi}\left[\frac{2\pi}{k} - \frac{\pi}{k}\right] = \frac{2}{k}$

$k = 1, 2, 3, \ldots$

$\therefore f(x) \simeq 0 + \sum_{k=1}^{\infty}\left(0 + b_k \sin kx\right) = \sum_{k=1}^{\infty} \frac{2}{k} \sin kx$

$\therefore \pi - x \; \{\text{over } x \in [0, 2\pi]\} \simeq \sum_{k=1}^{\infty} \frac{2}{k} \sin kx.$

$\frac{5c}{317}$ $y^2 - 8x - 14y + 49 = 0$

$\therefore (y-7)^2 - 49 - 8x + 49 = 0 \Rightarrow (y-7)^2 = 8x$

$\therefore y'^2 = 8x'$, parabola along x'

$\frac{5}{317}$ (d) $x^2 + y^2 + 6x - 10y + 18 = 0$

$\therefore (x+3)^2 - 9 + (y-5)^2 - 25 + 18 = 0$ or $x'^2 + y'^2 = 16$, Circle.

(e) $2x^2 - 3y^2 + 6x + 20y = -41$

$\therefore 2[x^2 + 3x] - 3[y^2 - \frac{20}{3}y] = -41 \Rightarrow 2\left[(x+\frac{3}{2})^2 - \frac{9}{4}\right] - 3\left[(y - \frac{10}{3})^2 - \frac{100}{9}\right] = -41$

$\therefore 2x'^2 - \frac{9}{2} - 3y'^2 + \frac{100}{3} = -41$ or $2x'^2 - 3y'^2 = -41 + \frac{9}{2} - \frac{100}{3} = \frac{-246+27-20}{6}$

$= -\frac{419}{6}$

$\therefore \frac{y'^2}{2} - \frac{x'^2}{3} = \frac{419}{36}$, Hyperbola.

(f) $x^2 + 10x + 7y = -32 \Rightarrow (x+5)^2 - 25 + 7y = -32$

$\therefore (x+5)^2 = -7y - 7 = -7(y+1) \quad \therefore x'^2 = -7y'$, parabola.

$\frac{6}{318}$ (a) $[x \;\; y]\begin{bmatrix}2 & -2\\ -2 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+8=0$ $\qquad A=\begin{bmatrix}2 & -2\\ -2 & -1\end{bmatrix}$ (Symmetric)

$\therefore |A-\lambda I| = \begin{vmatrix}2-\lambda & -2\\ -2 & -1-\lambda\end{vmatrix} = (2-\lambda)(-1-\lambda)-4 = -2-\lambda+\lambda^2-4 = \lambda^2-\lambda-6$
$= (\lambda-3)(\lambda+2) = 0 \qquad \therefore \lambda = 3 \text{ or } -2$

eign vectors for $\lambda=3$: $\begin{bmatrix}-1 & -2\\ -2 & -4\end{bmatrix} \Rightarrow x_1=-2x_2 \quad \therefore \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-2x_2\\ x_2\end{bmatrix}=-x_2\begin{bmatrix}2\\ -1\end{bmatrix}$

" " " $\lambda=-2$: $\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix} \Rightarrow 2x_1=x_2 \quad \therefore \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ 2x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ 2\end{bmatrix}$

$\therefore P=\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}\cdot\frac{1}{\sqrt5}$ is orthonormal (obtained from symmetric matrix with distinct roots) and $\det P=1$

Let $\begin{bmatrix}x\\ y\end{bmatrix}=P\begin{bmatrix}x'\\ y'\end{bmatrix} \quad \therefore [x \;\; y]=[x' \;\; y']P^T=[x' \;\; y']P^{-1}$

$\therefore [x \;\; y]A\begin{bmatrix}x\\ y\end{bmatrix}+8 = [x' \;\; y']P^{-1}AP\begin{bmatrix}x'\\ y'\end{bmatrix}+8 = [x' \;\; y']D\begin{bmatrix}x'\\ y'\end{bmatrix}+8=0$

and since $D=\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix} \quad \therefore [x' \;\; y']\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x'\\ y'\end{bmatrix}+8=0$

$\therefore 3x'^2-2y'^2=-8$ or $\frac{x'^2}{-8/3}-\frac{y'^2}{-8/2}=1 \Rightarrow \frac{y'^2}{4}-\frac{x'^2}{(8/3)}=1$, Hyperbola

(c) $[x \;\; y]\begin{bmatrix}5 & 2\\ 2 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=9 \qquad \therefore A=\begin{bmatrix}5 & 2\\ 2 & 5\end{bmatrix}$ (symmetric)

$|A-\lambda I| = \begin{vmatrix}5-\lambda & 2\\ 2 & 5-\lambda\end{vmatrix} = (5-\lambda)^2-4 = (5-\lambda-2)(5-\lambda+2)=(3-\lambda)(7-\lambda)=0$
$\therefore \lambda=3 \text{ or } 7$

eign vectors for $\lambda=3 \Rightarrow \begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix} \quad \therefore x_1=-x_2 \quad \therefore \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ -x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ -1\end{bmatrix}$

" " " $\lambda=7 \Rightarrow \begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix} \quad \therefore x_1=x_2 \quad \therefore \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ 1\end{bmatrix}$

$\therefore P=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\cdot\frac{1}{\sqrt2}$ is orthonormal and $\det P=1 \quad \therefore P^{-1}AP=D=\begin{bmatrix}3 & 0\\ 0 & 7\end{bmatrix}$

Let $\begin{bmatrix}x\\ y\end{bmatrix}=P\begin{bmatrix}x'\\ y'\end{bmatrix}$

$\therefore [x' \;\; y']D\begin{bmatrix}x'\\ y'\end{bmatrix}=9 \quad \therefore [x' \;\; y']\begin{bmatrix}3 & 0\\ 0 & 7\end{bmatrix}\begin{bmatrix}x'\\ y'\end{bmatrix}=3x'^2+7y'^2=9$

$\therefore \frac{x'^2}{3}+\frac{y'^2}{(9/7)}=1$, Ellipse

$\boxed{\frac{7}{318}}$ $[x\ \ y]\begin{bmatrix}9&-2\\-2&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+[-10\ \ -20]\begin{bmatrix}x\\y\end{bmatrix}=5$ (1) Let $A=\begin{bmatrix}9&-2\\-2&6\end{bmatrix}$

$\therefore |A-\lambda I| = \begin{vmatrix}9-\lambda&-2\\-2&6-\lambda\end{vmatrix} = (9-\lambda)(6-\lambda)-4 = 54-15\lambda+\lambda^2-4 = \lambda^2-15\lambda+50=0$

$\therefore (\lambda-10)(\lambda-5)=0 \quad \therefore \lambda = 5$ or 10

$\therefore$ Eign-vectors for $\lambda=5$ $\begin{bmatrix}4&-2\\-2&1\end{bmatrix}$ $\therefore 2x_1=x_2$ $\therefore \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=x_1\begin{bmatrix}1\\2\end{bmatrix}$

& " " $\lambda=10$ $\begin{bmatrix}-1&-2\\-2&-4\end{bmatrix}$ $\therefore x_1=-2x_2$ $\therefore \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2x_2\\x_2\end{bmatrix}=x_2\begin{bmatrix}-2\\1\end{bmatrix}$

$\therefore P=\frac{1}{\sqrt5}\begin{bmatrix}1&-2\\2&1\end{bmatrix}$ is orthonormal & $\det P=1$ Let $\begin{bmatrix}x\\y\end{bmatrix}=P\begin{bmatrix}x'\\y'\end{bmatrix}$

$\therefore$ Eq. (1) becomes $[x'\ \ y']\,P^{-1}AP\begin{bmatrix}x'\\y'\end{bmatrix}+[-10\ \ -20]\begin{bmatrix}1&-2\\2&1\end{bmatrix}\frac{1}{\sqrt5}\begin{bmatrix}x'\\y'\end{bmatrix}=5$

but $P^{-1}AP=D=\begin{bmatrix}5&0\\0&10\end{bmatrix}$ $\therefore 5x'^2+10y'^2+\frac{1}{\sqrt5}[-50\ \ 0]\begin{bmatrix}x'\\y'\end{bmatrix}=5$

or $5x'^2+10y'^2-\frac{50}{\sqrt5}x'=5$

$\therefore 5\left(x'^2-\frac{10}{\sqrt5}x'\right)+10y'^2=5 \quad \therefore 5\left[\left(x'-\frac{5}{\sqrt5}\right)^2-5\right]+10y'^2=5$

$\therefore (x'-\sqrt5)^2-5+2y'^2=1$ or $x''^2+2y''^2=6$, Ellipse

$\boxed{\frac{8}{318}}$

$$3x^2 - 8xy - 12y^2 - 30x - 64y = 0$$

$$\therefore \begin{bmatrix} x & y \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 3 & -4 \\ -4 & -12 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 30 & 64 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{V} = 0 \qquad (*)$$

$$\therefore P(\lambda) = |A - \lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ -4 & -12-\lambda \end{vmatrix} = (3-\lambda)(-12-\lambda) - 16 =$$

$$= -36 + 9\lambda + \lambda^2 - 16 = \lambda^2 + 9\lambda - 52 = (\lambda+13)(\lambda-4) = 0$$

$$\therefore \lambda = 4 \text{ or } -13$$

eign space for $\lambda = 4$: $\begin{bmatrix} 3-4 & -4 \\ -4 & -12-4 \end{bmatrix} \xrightarrow[4R_1 - R_2]{\frac{R_1}{-1}} \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \therefore x_1 + 4x_2 = 0$

$$\therefore x_1 = -4x_2 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_2 \\ x_2 \end{bmatrix} = -x_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$\therefore$ basis for eign space of $\lambda = 4$ is $P_1 = \langle 4, -1 \rangle / \sqrt{17}$

eign space for $\lambda = -13$: $\begin{bmatrix} 3+13 & -4 \\ -4 & -12+13 \end{bmatrix} \xrightarrow[\frac{R_1}{4} + R_2]{\frac{R_1}{4}} \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix} \therefore 4x_1 - x_2 = 0$

$$\therefore x_2 = 4x_1 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 4x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$\therefore$ basis for eign space of $\lambda = -13$ is $P_2 = \langle 1, 4 \rangle / \sqrt{17}$

$$\therefore P = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} / \sqrt{17} \quad \text{of} \quad P^{-1}AP = P^t AP = D = \begin{bmatrix} 4 & 0 \\ 0 & -13 \end{bmatrix}$$

Let $V = PV'$ $\therefore V^t A V = (PV')^t A (PV') = V'^t P^t A P V' = V'^t D V' =$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -13 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 4x'^2 - 13y'^2$$

$\therefore (*)$ becomes $4x'^2 - 13y'^2 - \begin{bmatrix} 30 & 64 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} / \sqrt{17} = 0$

or $4x'^2 - 13y'^2 - \frac{56}{\sqrt{17}} x' - \frac{286}{\sqrt{17}} y' = 0 \quad \therefore 4\underbrace{\left(x' - \frac{7}{\sqrt{17}}\right)}_{x''}^2 - \frac{196}{17} - 13\underbrace{\left(y' - \frac{11}{\sqrt{17}}\right)}_{y''}^2 + \frac{1573}{17} = 0$

$\therefore 4x''^2 - 13y''^2 + 81 = 0$

or $\left(\frac{y''}{9/\sqrt{13}}\right)^2 - \left(\frac{x''}{9/2}\right)^2 = 1$, hyperbola, y'' real axis.

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9/318 $\quad [x\ \ y]\begin{bmatrix}2&-2\\-2&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+[-4\ \ -8]\begin{bmatrix}x\\y\end{bmatrix}=-14$ (1) Let $A=\begin{bmatrix}2&-2\\-2&-1\end{bmatrix}$

$\therefore |A-\lambda I|=\begin{vmatrix}2-\lambda&-2\\-2&-1-\lambda\end{vmatrix}=(2-\lambda)(-1-\lambda)-4=-2-\lambda+\lambda^2-4=\lambda^2-\lambda-6=(\lambda-3)(\lambda+2)=0$ $\therefore \lambda=-2,3$

$\therefore P$ (as in $\frac{6a}{318}$) $=\begin{bmatrix}2&1\\-1&2\end{bmatrix}\frac{1}{\sqrt5}$ & $D=\begin{bmatrix}3&0\\0&-2\end{bmatrix}$ Let $\begin{bmatrix}x\\y\end{bmatrix}=P\begin{bmatrix}x'\\y'\end{bmatrix}$

$\therefore [x'\ \ y']\underbrace{P^{-1}AP}_{D}\begin{bmatrix}x'\\y'\end{bmatrix}+[-4\ \ -8]\frac{1}{\sqrt5}\begin{bmatrix}2&1\\-1&2\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix}=-14$

$\therefore 3x'^2-2y'^2+\frac{1}{\sqrt5}[0\ \ -20]\begin{bmatrix}x'\\y'\end{bmatrix}=-14 \Rightarrow 3x'^2-2y'^2-\frac{20}{\sqrt5}y'+14=0$

$\therefore 3x'^2-2[(y'+\sqrt5)^2-5]+14=0 \quad \therefore 3x''^2-2y''^2+24=0$, Hyperbola.

13/318 (e) $[x\ \ y]\begin{pmatrix}9&6\\6&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}-52=0$ (1) Let $A=\begin{bmatrix}9&6\\6&4\end{bmatrix}$

$|A-\lambda I|=\begin{vmatrix}9-\lambda&6\\6&4-\lambda\end{vmatrix}=36+\lambda^2-13\lambda-36=\lambda(\lambda-13)=0 \quad \therefore \lambda=0$ or 13

eign vectors for $\lambda=0$ $\begin{bmatrix}9&6\\6&4\end{bmatrix}\Rightarrow \therefore 3x_1+2x_2=0 \quad \therefore \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-\frac32 x_1\end{bmatrix}=\frac{x_1}{2}\begin{bmatrix}2\\-3\end{bmatrix}$

" " " $\lambda=13$ $\begin{bmatrix}-4&6\\6&-9\end{bmatrix}\Rightarrow -2x_1+3x_2=0 \quad \therefore \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\frac{3x_2}{2}\\x_2\end{bmatrix}=\frac{x_2}{2}\begin{bmatrix}3\\2\end{bmatrix}$

$\therefore P=\frac{1}{\sqrt{13}}\begin{bmatrix}2&3\\-3&2\end{bmatrix}$ & $P^{-1}AP=D=\begin{bmatrix}0&0\\0&13\end{bmatrix}$. Let $\begin{bmatrix}x\\y\end{bmatrix}=P\begin{bmatrix}x'\\y'\end{bmatrix}$ (2)

$\therefore$ Eq. (1) becomes:

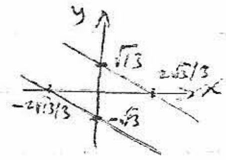
$[x'\ y']\,D\begin{bmatrix}x'\\y'\end{bmatrix}-52=0$ or $0x'^2+13y'^2-52=0$

$\therefore 13y'^2=52 \Rightarrow y'^2=4 \quad \therefore y'=\pm2$ Two straight ∥ lines

$\begin{bmatrix}x'\\y'\end{bmatrix}$ from (2) $=P^{-1}\begin{bmatrix}x\\y\end{bmatrix}=P^T$ (because P is orthonormal) $\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{\sqrt{13}}\begin{bmatrix}2&-3\\3&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$=\begin{bmatrix}2x-3y\\3x+2y\end{bmatrix}\frac{1}{\sqrt{13}} \quad \therefore y'=\frac{1}{\sqrt{13}}(3x+2y)$ but $y'=\pm2$

$\therefore \pm2=\frac{1}{\sqrt{13}}(3x+2y) \quad \therefore 3x+2y=\pm2\sqrt{13}$



(f) $x^2+y^2-2x-4y=-5$

$\therefore (x-1)^2\cancel{-1}+(y-2)^2\cancel{-4}=\cancel{-5} \quad \therefore (x-1)^2+(y-2)^2=0$

$\therefore x=1$ & $y=2$ $\therefore$ It is the point $(1,2)$.

5d/324 $\quad 4x^2+9y^2-z^2-54y-50z=544 \Rightarrow 4(x)^2+9(y-3)^2-(z+25)^2=0$

or $4x'^2+9y'^2=z'^2$ Elliptical cone along z.

$\boxed{\frac{6d}{324}}$ $2xy + z = 0 \therefore [x\ y\ z]\underbrace{\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}x\\ y\\ z\end{bmatrix} + [0\ 0\ 1]\underbrace{\begin{bmatrix}x\\ y\\ z\end{bmatrix}}_{V} = 0 \quad (*)$

$$P(\lambda) = \det(A - \lambda I)$$

$$= \begin{vmatrix}-\lambda & 1 & 0\\ 1 & -\lambda & 0\\ 0 & 0 & -\lambda\end{vmatrix} = -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1) = -\lambda(\lambda-1)(\lambda+1) = 0 \therefore \lambda = 0, \pm 1$$

eign space $\lambda = 0$: $\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \therefore x_2 = 0$ & $x_1 = 0 \therefore \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ x_3\end{pmatrix} = x_3\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} \therefore P_1 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

eign space $\lambda = 1$: $\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & -1\end{bmatrix} \xRightarrow{R_1 + R_2} \begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 0\\ 0 & 0 & -1\end{bmatrix} \therefore x_1 = x_2$ & $x_3 = 0 \therefore \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}x_1\\ x_1\\ 0\end{pmatrix} = x_1\begin{pmatrix}1\\ 1\\ 0\end{pmatrix} \therefore P_2 = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}/\sqrt{2}$

eign space $\lambda = -1$: $\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \Rightarrow x_1 + x_2 = 0$ & $x_3 = 0 \therefore \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}x_1\\ -x_1\\ 0\end{pmatrix} = x_1\begin{pmatrix}1\\ -1\\ 0\end{pmatrix} \therefore P_3 = \begin{pmatrix}1\\ -1\\ 0\end{pmatrix}/\sqrt{2}$

$$\therefore P = \begin{bmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 1 & 0 & 0\end{bmatrix} \quad \& \quad P^{-1}AP = P^tAP = D = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$\therefore V = PV' \quad \therefore V^tAV = (PV')^tAPV' = V'^tP^tAPV' = V'^tDV' = y'^2 - z'^2$

$\therefore (*)$ becomes $y'^2 - z'^2 + [0\ 0\ 1]\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 1 & 0 & 0\end{bmatrix}\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix} = y'^2 - z'^2 + [1\ 0\ 0]\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix} = 0$

OR $y'^2 - z'^2 + x' = 0$

$\therefore$ It is the surface of Hyperbolic Paraboloid

$\boxed{\frac{7}{324}}$ $2xy + 2xz + 2yz - 6x - 6y - 4z + 9 = 0$

$$\therefore \begin{bmatrix} x & y & z \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -6 & -6 & -4 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{V} + [9] = 0 \quad (*)$$

$$\therefore P(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 1 + 1 + \lambda + \lambda + \lambda = -\lambda^3 + 3\lambda + 2$$

$$= -\lambda^3 + \lambda + 2\lambda + 2 = -\lambda(\lambda^2 - 1) + 2(\lambda + 1)$$

$$= -\lambda(\lambda - 1)(\lambda + 1) + 2(\lambda + 1) = (\lambda + 1)[-\lambda(\lambda - 1) + 2] =$$

$$= (\lambda + 1)(-\lambda^2 + \lambda + 2) = -(\lambda + 1)(\lambda - 2)(\lambda + 1) = 0 \quad \therefore \lambda = 2, -1$$

Eign space for $\lambda = 2$:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_2 + R_3 + R_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{bmatrix} \therefore x_2 = x_3 \ \& \ x_1 = 2x_3 - x_2 = 2x_3 - x_3 = x_3 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ basis for eign space of $\lambda = 2$ is $P_1 = \langle 1, 1, 1 \rangle / \sqrt{3}$

Eign space for $\lambda = -1$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_1 - R_3]{R_1 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = -x_2 - x_3 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = -x_2 \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix} - x_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore$ basis for $\lambda = -1$ is $P_2 = \langle 1, -1, 0 \rangle / \sqrt{2}$

$$\& \ P_3 = \frac{\langle 1, 0, -1 \rangle - \langle 1, 0, -1 \rangle \cdot \langle 1, -1, 0 \rangle / \sqrt{2} * \langle 1, -1, 0 \rangle / \sqrt{2}}{|\quad '' \quad|} =$$

$$= \frac{\langle 1, 0, -1 \rangle - \frac{1}{2} \langle 1, -1, 0 \rangle}{|\quad '' \quad|} = \frac{\frac{1}{2} \langle 1, 1, -2 \rangle}{|\quad '' \quad|} = \langle 1, 1, -2 \rangle / \sqrt{6}$$

$$\therefore P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \ \& \ P^t A P = P^{-1} A P = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ Let } V = PV'$$

$$\therefore (*) \text{ becomes } (PV')^t A P V' + \begin{bmatrix} -6 & -6 & -4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} V' + [9] = 0$$

$$\therefore V'^T P^t A P V' + \begin{bmatrix} -\frac{16}{\sqrt{3}} & 0 & -\frac{4}{\sqrt{6}} \end{bmatrix} V' + [9] = 0$$

$$\therefore V'^T D V' + \begin{bmatrix} -\frac{16}{\sqrt{3}} & 0 & \frac{-4}{\sqrt{6}} \end{bmatrix} V' + [9] = 0 \quad \therefore 2x'^2 - y'^2 - z'^2 - \frac{16}{\sqrt{3}} x' - \frac{4}{\sqrt{6}} z' + 9 = 0$$

$$\therefore 2x'^2 - y'^2 - z'^2 - \frac{16x'}{\sqrt{3}} - \frac{4z'}{\sqrt{6}} + 9 = 0$$

$$\therefore 2\left(x' - \frac{4}{\sqrt{3}}\right)^2 - \frac{32}{3} - y'^2 - \left(z' + \frac{2}{\sqrt{6}}\right)^2 + \frac{4}{6} + 9 = 0$$

$$\therefore 2x''^2 - y''^2 - z''^2 + \frac{-64 + 4 + 54}{6} = 0, \text{ OR:}$$

$2x''^2 - y''^2 - z''^2 = 1$, <u>hyperbolid</u> of two sheets.

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$\boxed{\dfrac{8}{324}}$ $\quad [x\ \ y\ \ z]\underbrace{\begin{bmatrix}7&-1&-2\\-1&7&2\\-2&2&10\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}-12&12&60\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=24 \qquad (1)$

$$\therefore |A-\lambda I| = \begin{vmatrix}7-\lambda&-1&-2\\-1&7-\lambda&2\\-2&2&10-\lambda\end{vmatrix} = (7-\lambda)^2(10-\lambda)+4+4-8(7-\lambda)-(10-\lambda)$$
$$= (7-\lambda)[70+\lambda^2-17\lambda-8]+8-10+\lambda =$$
$$= (7-\lambda)(\lambda^2-17\lambda+62)+\lambda-2 = -\lambda^3+24\lambda^2-181\lambda+434+\lambda-2 =$$
$$= -\lambda^3+24\lambda^2-180\lambda+432 = -(\lambda-6)(\lambda^2-18\lambda+72) =$$
$$= -(\lambda-6)^2(\lambda-12)=0 \qquad \therefore \lambda = 6 \text{ or } 12$$

eigenvectors for $\lambda=6$: $\begin{bmatrix}1&-1&-2\\-1&+1&2\\-2&2&4\end{bmatrix}$ $\therefore x_1-x_2-2x_3=0$ $\therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\\frac{x_1-x_2}{2}\end{bmatrix}=\frac{x_1}{2}\underbrace{\begin{bmatrix}2\\0\\1\end{bmatrix}}_{u_1}+\frac{x_2}{2}\underbrace{\begin{bmatrix}0\\2\\-1\end{bmatrix}}_{u_2}$

$\therefore v_1 = \dfrac{u_1}{|u_1|} = \dfrac{\langle 2,0,1\rangle}{\sqrt5}$, $u_2 = c_1v_1+c_2v_2$; $u_2\cdot v_1 = c_1$ $\therefore c_1 = \langle 0,2,-1\rangle\cdot v_1$

$= \langle 0,2,-1\rangle\cdot\dfrac{\langle 2,0,1\rangle}{\sqrt5} = -\dfrac{1}{\sqrt5}$ $\therefore u_2 = -\dfrac{1}{\sqrt5}v_1 + c_2v_2$

$\therefore c_2v_2 = u_2+\dfrac{v_1}{\sqrt5} = \langle 0,2,-1\rangle + \dfrac15\langle 2,0,1\rangle = \langle \tfrac25, 2, -\tfrac45\rangle = \tfrac15\langle 2,10,-4\rangle$

$\therefore v_2 = \dfrac{\langle 2,10,-4\rangle}{\sqrt{4+100+16}} = \dfrac{\langle 2,10,-4\rangle}{\sqrt{120}} = \dfrac{\langle 1,5,-2\rangle}{\sqrt{30}}$

eigenvector for $\lambda=12$: $\begin{bmatrix}-5&-1&-2\\-1&-5&2\\-2&2&-2\end{bmatrix}\xrightarrow[R_3-2R_1]{R_1-5R_2}\begin{bmatrix}0&24&-12\\-1&-5&2\\0&12&-6\end{bmatrix}$ $\therefore 2x_2=x_3$ & $x_1 = 2x_3-5x_2 = 4x_2-5x_2 = -x_2$

$\therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_2\\x_2\\2x_2\end{bmatrix}=x_2\begin{bmatrix}-1\\1\\2\end{bmatrix}$ $\quad\therefore P=\begin{bmatrix}\frac{2}{\sqrt5}&\frac{1}{\sqrt{30}}&-1/\sqrt6\\0&5/\sqrt{30}&1/\sqrt6\\\frac{1}{\sqrt5}&-2/\sqrt{30}&2/\sqrt6\end{bmatrix}=\frac{1}{\sqrt{30}}\begin{bmatrix}2\sqrt6&1&-\sqrt5\\0&5&\sqrt5\\\sqrt6&-2&2\sqrt5\end{bmatrix}$

is orthonormal with $\det P=1$ & $P^{-1}AP = D = \begin{bmatrix}6&0&0\\0&6&0\\0&0&12\end{bmatrix}$

Let $\begin{bmatrix}x\\y\\z\end{bmatrix}=P\begin{bmatrix}x'\\y'\\z'\end{bmatrix}$

$\therefore$ (1) becomes $[x'\ \ y'\ \ z']\,D\begin{bmatrix}x'\\y'\\z'\end{bmatrix}+12\begin{bmatrix}-1&1&5\end{bmatrix}\begin{bmatrix}2\sqrt6&1&-\sqrt5\\0&5&\sqrt5\\\sqrt6&-2&2\sqrt5\end{bmatrix}\cdot\frac{1}{\sqrt{30}}\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=24$

$\therefore 6x'^2+6y'^2+12z'^2+\dfrac{12}{\sqrt{30}}\begin{bmatrix}3\sqrt6&-6&12\sqrt5\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=24$

$$\therefore x'^2+y'^2+2z'^2+\frac{6}{\sqrt5}x'-\frac{12}{\sqrt{30}}y'+\frac{24}{\sqrt6}z' = 4$$

$$\therefore \left(x'+\frac{3}{\sqrt5}\right)^2-\frac95+\left(y'-\frac{6}{\sqrt{30}}\right)^2-\frac{36}{30}+2\left[\left(z'+\frac{6}{\sqrt6}\right)^2-6\right]=4$$
$$\therefore x''^2+y''^2+2z''^2 = \frac95+\frac{36}{30}+12+4 = 19$$, Ellipsoid.

$\boxed{\frac{9}{324}}$ $2xy - 6x + 10y + z - 31 = 0 \quad \therefore \quad [x\ y\ z]\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} + [-6\ \ 10\ \ 1]\begin{bmatrix}x\\y\\z\end{bmatrix} - 31 = 0 \quad (1)$

Let $A = \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \quad \therefore \quad |A - \lambda I| = \begin{vmatrix}-\lambda&1&0\\1&-\lambda&0\\0&0&-\lambda\end{vmatrix} = -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1) = -\lambda(\lambda - 1)(\lambda + 1) = 0$

$\therefore \lambda = 0, 1, -1$

eign-vectors for $\lambda = 0$: $\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \Rightarrow x_1 = x_2 = 0 \quad \therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\x_3\end{bmatrix} = x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$.

" " " $\lambda = 1$: $\begin{bmatrix}-1&1&0\\1&-1&0\\0&0&-1\end{bmatrix} \Rightarrow x_1 = x_2 \ \& \ x_3 = 0 \quad \therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_1\\0\end{bmatrix} = x_1\begin{bmatrix}1\\1\\0\end{bmatrix}$.

eign-vectors for $\lambda = -1$: $\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix} \quad \therefore \quad x_1 = -x_2 \ \& \ x_3 = 0 \quad \therefore \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{bmatrix}-x_2\\x_2\\0\end{bmatrix} = x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$\therefore \quad P = \begin{bmatrix}0&\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\1&0&0\end{bmatrix} = \frac{1}{\sqrt2}\begin{bmatrix}0&1&-1\\0&1&1\\\sqrt2&0&0\end{bmatrix}$ (orthonormal with $\det P = 1$)

Let $\begin{bmatrix}x\\y\\z\end{bmatrix} = P\begin{bmatrix}x'\\y'\\z'\end{bmatrix}$ in (1) $\Rightarrow [x'\ y'\ z']\,P^TAP\begin{bmatrix}x'\\y'\\z'\end{bmatrix} + [-6\ \ 10\ \ 1]\begin{bmatrix}0&1&-1\\0&1&1\\\sqrt2&0&0\end{bmatrix}\frac{1}{\sqrt2}\begin{bmatrix}x'\\y'\\z'\end{bmatrix} - 31 = 0$

but $P^TAP = P^{-1}AP$ (P orthonormal $\therefore P^T = P^{-1}$) $= D = \begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}$

$\therefore [x'\ y'\ z']\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\end{bmatrix} + [\sqrt2\ \ 4\ \ 16]\begin{bmatrix}x'\\y'\\z'\end{bmatrix} - 31 = 0$

$\therefore \quad y'^2 - z'^2 + \sqrt2 x' + 4y' + 16z' - 31 = 0$

$\therefore \quad (y'+2)^2 - 4 - (z'-8)^2 + 64 + \sqrt2 x' - 31 = 0$

$\therefore \quad (y'+2)^2 - (z'-8)^2 + \sqrt2\left(x' + \frac{29}{\sqrt2}\right) = 0$

$\therefore \quad y''^2 - z''^2 + \sqrt2 x'' = 0$

or $\sqrt2 x'' = z''^2 - y''^2$, Hyperbolic Paraboloid.

$\boxed{\frac{10}{324}}$ $[x\ y\ z]\underbrace{\begin{bmatrix}2&-2&-1\\-2&2&1\\-1&1&5\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\end{bmatrix}+[10\ \ -26\ \ -2]\begin{bmatrix}x\\y\\z\end{bmatrix}=0$ (1) $\therefore |A-\lambda I|=\begin{vmatrix}2-\lambda&-2&-1\\-2&2-\lambda&1\\-1&1&5-\lambda\end{vmatrix}=$

$=(2-\lambda)^2(5-\lambda)+4-2(2-\lambda)-4(5-\lambda)=(2-\lambda)(10+\lambda^2-7\lambda-2)+4-20+4\lambda=$

$=(2-\lambda)(8-7\lambda+\lambda^2)+4\lambda-16=16-22\lambda+9\lambda^2-\lambda^3+4\lambda-16=-\lambda(\lambda^2-9\lambda+18)=$

$=-\lambda(\lambda-3)(\lambda-6)=0 \quad \therefore \lambda=0$ or 3 or 6

eigenvectors for $\lambda=0$ $\begin{bmatrix}2&-2&-1\\-2&2&1\\-1&1&5\end{bmatrix}\xrightarrow{\substack{R_1+R_2\\R_1+2R_3}}\begin{bmatrix}2&-2&-1\\0&0&0\\0&0&9\end{bmatrix}$ $\therefore x_3=0$ & $x_1=x_2$ $\therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_2\\x_2\\0\end{bmatrix}=x_2\begin{bmatrix}1\\1\\0\end{bmatrix}$

" " $\lambda=3$:

$\therefore \begin{bmatrix}-1&-2&-1\\-2&-1&1\\-1&1&2\end{bmatrix}\xrightarrow{\substack{-R_1\\R_2-2R_1\\R_3-R_1}}\begin{bmatrix}1&2&1\\0&3&3\\0&3&3\end{bmatrix}$ $\therefore x_2=-x_3$ & $x_1=-x_3-2x_2=-x_3+2x_3=x_3$ $\therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\-1\\1\end{bmatrix}$

eigenvectors for $\lambda=6$: $\begin{bmatrix}-4&-2&-1\\-2&-4&1\\-1&1&-1\end{bmatrix}\xrightarrow{\substack{R_1-4R_3\\R_2-2R_3}}\begin{bmatrix}0&-6&3\\0&-6&3\\-1&1&-1\end{bmatrix}$ $\therefore x_3=2x_2$ & $x_1=x_2-x_3=x_2-2x_2=-x_2$ $\therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=-x_2\begin{bmatrix}1\\-1\\-2\end{bmatrix}$

$\therefore P=\begin{bmatrix}1/\sqrt2&1/\sqrt3&+1/\sqrt6\\+1/\sqrt2&-1/\sqrt3&-1/\sqrt6\\0&1/\sqrt3&-2/\sqrt6\end{bmatrix}=\frac{1}{\sqrt6}\begin{bmatrix}\sqrt3&\sqrt2&1\\\sqrt3&-\sqrt2&-1\\0&\sqrt2&-2\end{bmatrix}$ & $P^{-1}AP=D=\begin{bmatrix}0&0&0\\0&3&0\\0&0&6\end{bmatrix}$. Let $\begin{bmatrix}x\\y\\z\end{bmatrix}=P\begin{bmatrix}x'\\y'\\z'\end{bmatrix}$

$\therefore$ (1) becomes $0x'^2+3y'^2+6z'^2+[10\ \ -26\ \ -2]\frac{1}{\sqrt6}\begin{bmatrix}\sqrt3&\sqrt2&+1\\\sqrt3&-\sqrt2&-1\\0&\sqrt2&-2\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=0$

$\therefore 3y'^2+6z'^2+\frac{1}{\sqrt6}[-16\sqrt3\ \ 34\sqrt2\ \ +40]\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=0 \quad \therefore 3y'^2+6z'^2-\frac{16}{\sqrt2}x'+\frac{34}{\sqrt3}y'+\frac{40}{\sqrt6}z'=0$

$\therefore 3\left[\left(y'+\frac{17}{3\sqrt3}\right)^2-\frac{289}{27}\right]+6\left[\left(z'+\frac{20}{6\sqrt6}\right)^2-\frac{100}{54}\right]-\frac{16}{\sqrt2}x'=0 \quad \therefore 3y''^2+6z''^2=\frac{16}{\sqrt2}x'+\frac{289}{9}+\frac{100}{9}$

$=\frac{16}{\sqrt2}x'+\frac{389}{9}=\frac{16}{\sqrt2}\left(x'+\frac{389\sqrt2}{144}\right)=\frac{16}{\sqrt2}x'' \quad \therefore 3y''^2+6z''^2=\frac{16}{\sqrt2}x''$, Elliptical Paraboloid.

$\boxed{\frac{3}{425}}$ $u_{tt} + 2b\,u_t = a^2 u_{xx}$ Let $u(t,x) = g(t)\cdot f(x)$

$\therefore g''\cdot f + 2b\,g'\cdot f = a^2 g\cdot f''$ dividing by $g\cdot f$

$\therefore \frac{g''}{g} + 2\frac{bg'}{g} = a^2\frac{f''}{f}$ (and this must equals constant) $= c$

$\therefore a^2 f'' = cf \Rightarrow (a^2D^2 - c)f = 0 \quad \therefore f = \begin{cases} c<0: & f(x) = B_1\cos kx + B_2\sin kx \quad \text{where } k = \sqrt{\frac{-c}{a^2}} = \frac{\sqrt{-c}}{a} \\ c=0: & f(x) = B_3 + B_4 x \\ c>0: & f(x) = B_5 e^{k'x} + B_6 e^{-k'x} \quad \text{where } k' = \sqrt{\frac{c}{a^2}} = \frac{\sqrt{c}}{a} \end{cases}$

f $g'' + 2bg' = cg$

$\therefore (D^2 + 2bD - c)g = 0 \qquad \therefore D = \frac{-2b \pm \sqrt{4b^2+4c}}{2} = -b \pm \sqrt{b^2+c}$

and since c has three cases, it can be seen that D has always real roots for $c \geq 0$ and for $c<0$ roots can be real or imaginary

$$\therefore g(t) = \begin{cases} c<0: & \begin{cases} b^2+c>0=\gamma^2: & A_1 e^{(-b+\gamma)t} + A_2 e^{(-b-\gamma)t} = e^{-bt}(A_1 e^{\gamma t} + A_2 e^{-\gamma t}) \\ b^2+c<0=-\delta^2: & e^{-bt}(A_3\cos\delta t + A_4 \sin\delta t) \\ b^2+c=0: & A_5 e^{-bt} + A_6 t e^{-bt} = e^{-bt}(A_5 + A_6 t) \end{cases} \\ c>0: & b^2+c>0=\gamma^2 \text{ as in the first case above.} \end{cases}$$

$\therefore$ The solution is

$u(t,x) = g(t)\cdot f(x) = e^{-bt}(A_1 e^{\gamma t} + A_2 e^{-\gamma t})(B_1\cos kx + B_2\sin kx)$; $c<0$ & $b^2+c=\gamma^2>0$

$= e^{-bt}(A_3\cos\delta t + A_4\sin\delta t)(B_1\cos kx + B_2\sin kx)$: $c<0$ & $b^2+c=-\delta^2<0$

$= e^{-bt}(A_5 + A_6\cdot t)(B_1\cos kx + B_2\sin kx)$: $c<0$ & $b^2+c=0$

$= e^{-bt}\cdot(A_1 e^{\gamma t} + A_2 e^{-\gamma t})(B_3 + B_4 x)$: $c=0$, $\gamma=b$

$= e^{-bt}(A_1 e^{\gamma t} + A_2 e^{-\gamma t})(B_5 e^{k'x} + B_6 e^{-k'x})$: $c>0$ & $b^2+c=\gamma^2>0$, $k'=\sqrt{\frac{c}{a^2}}$

$\boxed{\frac{4}{425}}$ $w_y = y\, w_x$, Let $w = f(x) \cdot g(y)$

$\therefore f g' = y f' g$ dividing by $f \cdot g$

$\therefore \frac{g'}{g} = \frac{y f'}{f}$ or $\frac{g'}{y g} = \frac{f'}{f} = c$

$\therefore f' - cf = 0 \Rightarrow (D - c) f = 0 \quad \therefore f = A_1 e^{cx}$

& $\frac{g'}{g} = cy \Rightarrow \frac{dg}{g\, dy} = cy \Rightarrow \int \frac{dg}{g} = \int cy\, dy \quad \therefore \ln g = \frac{cy^2}{2} + A_2$

$\therefore g = e^{\frac{cy^2}{2} + A_2} = A_3 e^{\frac{cy^2}{2}}$

$\therefore w = f \cdot g = A_1 e^{cx} * A_3 e^{\frac{cy^2}{2}} = A_4 e^{c(x + \frac{y^2}{2})}$ is the solution.

$\boxed{\frac{5}{426}}$ $x w_x = w + y w_y$ Let $w(x,y) = g(x) \cdot f(y)$

$\therefore x g' f = g f + y \cdot g f'$ dividing by $w = g \cdot f$

$\therefore \frac{x g'}{g} = 1 + \frac{y f'}{f} = (\text{must})\ c$

$\therefore \frac{x g'}{g} = c \quad \therefore \frac{g'}{g} = \frac{c}{x} \Rightarrow \frac{dg/dx}{g} = \frac{c}{x} \Rightarrow \int \frac{dg}{g} = \int \frac{c\, dx}{x}$

$\therefore \ln g = c \ln x + A = \ln x^c + A$

$\therefore g = e^{\ln x^c + A} = e^A \cdot x^c = B x^c$

& $1 + \frac{y f'}{f} = c \quad \therefore y f' + (1 - c) f = 0 \quad \therefore \frac{f'}{f} = \frac{c-1}{y} \Rightarrow \frac{df}{f\, dy} = \frac{c-1}{y}$

$\therefore \int \frac{df}{f} = \int (c-1) \frac{dy}{y} \quad \therefore \ln f = (c-1) \ln y + A' = \ln y^{c-1} + A'$

$\therefore f = e^{\ln y^{c-1} + A'} = e^{A'} \cdot y^{c-1} = B' y^{c-1}$

$\therefore u = g f = B x^c \cdot B' y^{c-1} = C' x^c y^{c-1}$ where c, C' are constants

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$\boxed{\frac{10}{426}}$ $u_{xx} + 2 u_{xt} + u_{tt} = 0$, $u = e^{\beta t} g(x)$

$\therefore e^{\beta t} g'' + 2\beta e^{\beta t} g' + \beta^2 e^{\beta t} g = 0$ dividing by

$\therefore g'' + 2\beta g' + \beta^2 g = 0 \quad \therefore (D^2 + 2\beta D + \beta^2) g = 0$

$\therefore D = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\beta^2}}{2} = -\beta \pm 0 \quad \therefore g = A_1 e^{-\beta x} + A_2 x e^{-\beta x} = e^{-\beta x}(A_1 + A_2 x)$

$\therefore u = e^{\beta t} \cdot g$

$= e^{\beta t} \cdot e^{-\beta x} \cdot (A_1 + A_2 x) = e^{\beta(t - x)} \cdot (A_1 + A_2 x)$

$\boxed{\frac{12}{4\text{-}27}}$ $V = f(xy)$ Solves $x V_x = y V_y$

$LHS = x V_x = x f_x = x f' \cdot y = xy f'$
$RHS = y V_y = y f_y = y f' x = xy f'$

$\therefore$ $RHS = LHS$ $\therefore$ $f(xy) = V$ is a solution for $xV_x = yV_y$

$\boxed{\frac{13}{4\cdot 27}}$ $w_y = y w_x$ $w = f(2x + y^2)$

$LHS = w_y = f' \cdot 2y = 2y f'$
$RHS = y w_x = y \cdot f' \cdot 2 = 2y f' = LHS$
$\therefore$ $w = f(2x + y^2)$ is a solution for $w_y = y w_x$

w found to be in $\boxed{\frac{4}{425}}$ as $w(x,y) = A e^{c(2x+y^2)}$ → A, C are constants

$\therefore$ $f(2x + y^2) = A e^{c(2x+y^2)}$ $\therefore$ $f(z) = A e^{cz}$

$\boxed{\frac{3}{473}}$ $u_t = h^2 u_{xx}$ since u is independent of t $\therefore u_t = 0$
$\therefore$ $0 = h^2 u_{xx} = h^2 u''$ $\therefore u'' = 0$ $\therefore u = C_1 + C_2 x$
$u(0) = A$ $\therefore C_1 = A$ $u(c) = 0$ $\therefore C_1 + C_2 c = 0$ $\therefore C_2 = \frac{-C_1}{c} = \frac{-A}{c}$
$\therefore$ $u = A - \frac{A}{c} x = A\left(1 - \frac{x}{c}\right)$
$\therefore$ The solution is $u(x) = A\left(1 - \frac{x}{c}\right)$

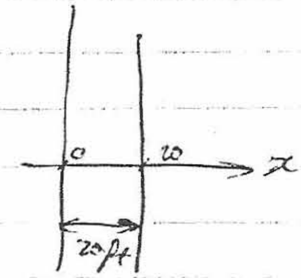
$\boxed{\frac{6}{473}}$

$h^2\pi^2 = 0.4$

$u(0,x) = 130\,°F$

$u(t,0) = u(t,20) = 60\,°F,\ t>0$

$u_t = h^2 u_{xx}$, Let $u(t,x) = f(t)\,g(x)$



∴ $f'g = h^2 f g'' \Rightarrow \frac{f'}{f} = \frac{h^2 g''}{g} = c$

∴ $f = Ae^{ct}$

& $g = \begin{cases} B_1x + B_2 & c=0 \\ B_3e^{kx/h} + B_4e^{-kx/h} & 0<c=k^2 \\ B_5\sin\left(\frac{kx}{h}\right) + B_6\cos\left(\frac{kx}{h}\right) & 0>c=-k^2 \end{cases}$

for $c=0$

∴ $u(t,x) = f\cdot g = A_1x + A_2$ ∴ $u(t,0) = A_2 = 60$ & $u(t,20) = 20A_1 + A_2 = 60$ ∴ $20A_1 + 60 = 60$ ∴ $A_1 = 0$ ∴ $u(t,x) = 60\,°F$

for $0<c=k^2$ ∴ $\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}u_t(t,0)\\u_t(t,20)\end{bmatrix} = cAe^{ct}\begin{bmatrix}1 & 1\\ e^{20k/h} & e^{-20k/h}\end{bmatrix}\begin{bmatrix}B_3\\B_4\end{bmatrix}$ ∴ $-2\sinh(20k/h) = \Delta$

$\Delta \neq 0 \Rightarrow B_3 = B_4 = 0$ ×

$\Delta = 0 \Rightarrow k = 0$ ×

for $0>c=-k^2$ ∴ $\begin{bmatrix}0\\0\end{bmatrix} = cAe^{ct}\begin{bmatrix}0 & 1\\ \sin\frac{20k}{h} & \cos\frac{20k}{h}\end{bmatrix}\begin{bmatrix}B_5\\B_6\end{bmatrix}$

$\Delta \neq 0 \Rightarrow B_5 = B_6 = 0$

$\Delta = 0 \Rightarrow \sin\frac{20k}{h} = 0$ & $B_6 = 0$

∴ $\frac{20k}{h} = n\pi$ ∴ $k = \frac{n\pi h}{20}$, n integer

∴ $u(t,x) = 60 + \sum_{n=0}^{\infty} Ae^{ct}\cdot B_5 \sin\frac{kx}{h} = 60 + \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi h}{20}\right)^2 t}\sin\left(\frac{n\pi x}{20}\right)$

at $t=0$ $u(0,x) = 130 = 60 + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2h^2}{400}\cdot 0}\cdot \sin\frac{n\pi x}{20}$

∴ $70 = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{20}$ ∴ $A_n = \frac{2}{20}\int_0^{20} 70 \sin\frac{n\pi x}{20}\,dx = 7\,\frac{\cos(n\pi x/20)}{-n\pi/20}\Big|_0^{20} =$

$= -\frac{140}{n\pi}(\cos n\pi - 1) = 0$ for n even & $\frac{280}{n\pi}$ for n odd

∴ $u(t,x) = 60 + \sum_{n\ \text{odd}} \frac{280}{n\pi} e^{-\frac{n^2 t}{1000}} \sin\left(\frac{n\pi x}{20}\right)$

at mid-thickness ∴ $x = 10$ ft, put $n = 2m+1$:

∴ $u(t,10) = 60 + \underline{\frac{280}{\pi}} \sum_{m=0}^{\infty} e^{-\frac{(2m+1)^2 t}{1000}} \cdot \left[\frac{(-1)^m}{2m+1}\right]$.

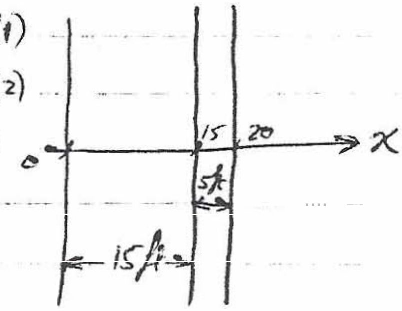
$\boxed{\frac{7}{473}}$

$u(0,x) = 120\,°F \qquad x \in (0,15) \quad (1)$

$\qquad\quad = 30\,°F \qquad x \in (15,20) \quad (2)$

$u(t,0) = u(t,20) = 30\,°F \qquad t>0 \quad (3)$

$u_t(t,0) = u_t(t,20) = 0 \qquad t>0 \quad (4)$



Let $u(t,x) = f(t)\cdot g(x)$

$\therefore \; u_t = h^2 u_{xx} \Rightarrow f'g = h^2 f g'' \Rightarrow \frac{f'}{f} = \frac{h^2 g''}{g} = c$

$\therefore \; \frac{f'}{f} = c \Rightarrow f(t) = Ae^{ct}$

& $g'' = \frac{c}{h^2} g \Rightarrow g(x) = \begin{cases} c=0: & B_1 x + B_2 \\ 0<c=k^2: & B_3 e^{kx/h} + B_4 e^{-kx/h} \\ 0>c=-k^2: & B_5 \cos\left(\frac{kx}{h}\right) + B_6 \sin\left(\frac{kx}{h}\right) \end{cases}$

where $A_i = AB_i$

$\therefore \; u(t,x) = fg = \begin{cases} c=0: & A_1 x + A_2 \\ 0<c=k^2: & e^{k^2 t}\cdot\left(A_3 e^{kx/h} + A_4 e^{-kx/h}\right) \\ 0>c=-k^2: & e^{-k^2 t}\cdot\left(A_5 \cos\left(\frac{kx}{h}\right) + A_6 \sin\left(\frac{kx}{h}\right)\right) \end{cases}$

for $c=0$ apply (4) to the solution:

$\therefore \; 0 = 0 \qquad \therefore$ OK

applying (3)

$\therefore \; 30 = A_1 * 0 + A_2 = A_1 * 20 + A_2 \qquad \therefore \begin{bmatrix} 30 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 20 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \therefore \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -1 \\ -20 & 0 \end{bmatrix}\begin{bmatrix} 30 \\ 30 \end{bmatrix}}{-20}$

$\therefore \; A_2 = 30\,°F$ & $A_1 = 0 \qquad \therefore \; u(t,x) = 30\,°F$ is a solution.

for $0 < c = k^2$: apply (4) to the solution:

$\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} = k^2 e^{k^2 t} \cdot \begin{bmatrix} 1 & 1 \\ e^{20k/h} & e^{-20k/h} \end{bmatrix} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} \qquad \therefore \; \Delta = e^{-20k/h} - e^{20k/h} = -2\sinh(20k/h)$

$\because \; k^2 > 0 \;\therefore\; k \neq 0 \;\therefore\; \Delta \neq 0$

$\therefore$ Only trivial solution $\therefore A_3 = A_4 = 0$

for $0 > c = -k^2$: apply (4) to the solution:

$\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -k^2 e^{-k^2 t} \cdot \begin{bmatrix} 1 & 0 \\ \cos(20k/h) & \sin(20k/h) \end{bmatrix} \begin{bmatrix} A_5 \\ A_6 \end{bmatrix} \Rightarrow \Delta = \sin(20k/h)$

$\therefore$ either $\Delta \neq 0 \;\therefore A_5 = A_6 = 0$

OR $\Delta = 0$ i.e. $\frac{20k}{h} = n\pi \quad n = 0, \pm 1, \ldots$

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$\therefore\ k = \frac{n\pi h}{20}$, n = integer

$\therefore\ A_5 = 0$ & A_6 is arbitrary constant

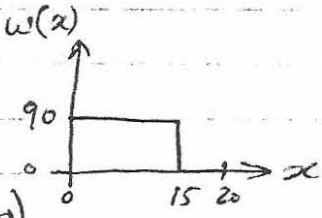
$\therefore$ general solution is $u(t,x) = 30 + \sum_n e^{-\left(\frac{n\pi h}{20}\right)^2 t} \cdot A_{6_n} \sin\left(\frac{n\pi x}{20}\right)$, n integer

$$= 30 + \sum_{n=1}^{\infty} A_n \cdot e^{-\left(\frac{n\pi h}{20}\right)^2 t} \sin\left(\frac{n\pi x}{20}\right)$$

Applying (1) & (2)

$$\therefore \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{20}\right) = w(x) = 90°F \quad x \in (0,15)$$
$$= 0\,°F \quad x \in (15,20)$$

$$\therefore A_n = \frac{2}{20}\int_0^{20} w(x) \cdot \sin\left(\frac{n\pi x}{20}\right) dx =$$



$$= \frac{1}{10}\int_0^{15} 90 \sin\left(\frac{n\pi x}{20}\right) dx = 9 \cdot \frac{\cos\left(\frac{n\pi x}{20}\right)}{-\left(\frac{n\pi}{20}\right)}\Bigg|_0^{15}$$

$$= -\frac{180}{n\pi}\left(\cos\left(\frac{15 n\pi}{20}\right) - 1\right) = \frac{180}{n\pi}\left(1 - \cos\left(\frac{3n\pi}{4}\right)\right)$$

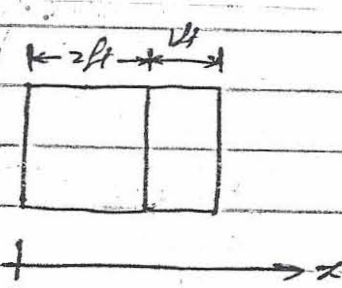
$$\therefore u(t,x) = 30 + \sum_{n=1}^{\infty} \frac{180}{n\pi}\left(1 - \cos\frac{3n\pi}{4}\right) \cdot e^{-\frac{n^2\pi^2 h^2}{400} t} \sin\frac{n\pi x}{20}$$

$\because\ h^2\pi^2 = 0.4$

$$\therefore u(t,x) = 30 + \frac{180}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1 - \cos\left(\frac{3n\pi}{4}\right)}{n} \cdot e^{-\frac{n^2 t}{1000}} \cdot \sin\left(\frac{n\pi x}{20}\right)$$

$\boxed{\frac{8}{473}}$ $u_t = h^2 u_{xx}$

since the same material $\therefore$ h is same for both



boundry conditions:

$u = A \quad t = 0 \quad 0 \le x \le 2$

$u = 0 \quad t = 0 \quad 2 < x \le 3$

$u = 0 \quad$ at $t > 0 \quad x = 0 \quad \therefore u(t,0) = 0$

$u = 0 \quad$ at $t > 0 \quad x = 3 \quad \therefore u(t,3) = 0$

Let $u = f(t) \cdot g(x)$

$$\therefore f'g = h^2 f g'' \implies \frac{f'}{f} = h^2 \frac{g''}{g} = c$$

$$\therefore f' = cf \implies f = B_1 e^{ct}$$

& $h^2 g'' = cg \quad \therefore (h^2 D^2 - c) g = 0 \quad \therefore g = \begin{cases} B_2 \sin kx + B_3 \cos kx,\ k^2 = \frac{-c}{h^2} & c<0 \\ B_4 + B_5 x & c=0 \\ B_6 e^{k'x} + B_7 e^{-k'x},\ k'^2 = \frac{c}{h^2} & c>0 \end{cases}$$

$$\therefore\ u(t,x) = fg = \begin{cases} c<0: & e^{ct}(B_2'\sin kx + B_3'\cos kx) \\ c=0: & e^{ct}(B_4' + B_5' x) \\ c>0: & e^{ct}(B_6' e^{k'x} + B_7' e^{-k'x}) \end{cases}$$

$$u(t,0) = u(t,3) = 0 \implies \begin{bmatrix}0\\0\end{bmatrix} = \begin{cases} c<0: & e^{ct}\begin{bmatrix}B_3' \\ B_2'\sin 3k + B_3'\cos 3k\end{bmatrix} \\ c=0: & e^{ct}\begin{bmatrix}B_4' \\ B_4' + 3B_5'\end{bmatrix} \therefore B_4' = B_5' = 0 \ \therefore \times \\ c>0: & e^{ct}\begin{bmatrix}B_6' + B_7' \\ B_6' e^{3k'} + B_7' e^{-3k'}\end{bmatrix} \therefore B_6' = B_7' = 0 \ \therefore \times \end{cases}$$

$\therefore$ c must be negative to have a solution

$\therefore$ $B_3' = 0$ & $B_2' \sin 3k = 0$ $\therefore$ $3k = n\pi$, $n = 0, 1, 2, \ldots$ integer

$\therefore$ $k = \frac{n\pi}{3}$, $\frac{-c}{h^2} = k^2 = \frac{n^2\pi^2}{9}$ $\therefore$ $c = -\frac{n^2\pi^2 h^2}{9}$ (negative)

$$\therefore\ u(t,x) = e^{ct}(B_2'\sin kx + \cancel{B_3'}\cos kx) = B_2' e^{ct} \sin \frac{n\pi}{3} x,\ n = 1, 2, \ldots$$

$$\therefore\ u(t,x) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2h^2}{9}t} \cdot \sin \frac{n\pi x}{3}$$

$$\text{But } u(0,x) = \left.\begin{cases} A & x \in [0,2] \\ 0 & x \in (2,3] \end{cases}\right\} = w(x)$$

$$\therefore\ w(x) = u(0,x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{3}$$

$\therefore$ By odd extension of Fourier Series (sine coeffecient $= \frac{4}{P} \int_0^{P/2} w(x) \sin \frac{n\pi x}{3} dx$, P = period at $n=1$)

$$\therefore\ B_n = \frac{4}{6}\int_0^3 w(x) \sin\frac{n\pi x}{3}dx = \frac{2}{3}\left[\int_0^2 A \sin\frac{n\pi x}{3}dx + \int_2^3 0 \sin\frac{n\pi x}{3}dx\right] = \frac{2A}{3}\left[\frac{-\cos n\pi x/3}{n\pi/3}\right]_0^2 =$$

$$= \frac{2A}{\cancel{3}}\cdot\frac{\cancel{3}}{n\pi}\left[-\cos\frac{2n\pi}{3} + 1\right] = \frac{2A}{n\pi}\cdot\frac{3}{2},\ n = 1, 2, 4, 5, \ldots \text{ not divisible by 3.}$$
$$= 0,\ n \text{ divisible by 3.}$$

$\therefore$ $B_n = \frac{3A}{n\pi}$ where n is integer not dividing by 3.

$$\therefore\ u(t,x) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2h^2t}{9}} \sin\frac{n\pi x}{3} =$$

$$= \sum_{\substack{n=1 \\ n \neq 3, 6, \ldots 3K,\ K \text{ integer}}}^{\infty} \frac{3A}{n\pi} e^{-\frac{n^2\pi^2h^2}{9}\cdot t} \cdot \sin\frac{n\pi x}{3}$$

$\therefore$ Temperature at centre of 2 ft-slab $= u(t,1) =$

$$= \sum_{\substack{n=1 \\ n \neq 3K,\ K \text{ integer}}}^{\infty} \frac{3A}{n\pi} e^{-\left(\frac{n\pi h}{3}\right)^2 t} \cdot \sin\frac{n\pi}{3} = \sum_{\substack{n=1 \\ \text{All } n}}^{\infty} \frac{3A}{n\pi} e^{-\left(\frac{n\pi h}{3}\right)^2 t} \sin\left(\frac{n\pi}{3}\right)$$

(because $\sin\frac{n\pi}{3} = 0$ anyway for $n = 3K$)

$\therefore$ The temperature at mid 2 ft-slab is $u(t,1) = \frac{3A}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi/3)}{n} \cdot e^{-\left(\frac{n\pi h}{3}\right)^2 t}$.

$\boxed{\frac{1}{480}}$ $= \quad {}^2 \quad + \frac{2}{\rho} u_\rho) \quad t> \quad \rho \in [0, K]$ (4) pp479

$u = f(\rho)$ at $t = 0$, $\rho \in [0, R]$ (6) pp479

$u = 0$ at $t > 0$, $\rho \equiv R^- \therefore u(t, R) = 0$ (5) pp479

Let $u(t, \rho) = g(t)\, w(\rho)$

$\therefore g' w = h^2 \left(g w'' + \frac{2}{\rho} g \cdot w' \right)$ dividing by $u = gw$

$\therefore \frac{g'}{g} = h^2 \left(\frac{w''}{w} + \frac{2}{\rho} \frac{w'}{w} \right) = c$

$g' = cg \quad \therefore g(t) = A e^{ct}$

& $h^2 \left(w'' + \frac{2}{\rho} w' \right) = c w$ put $w = \frac{v}{\rho}$, $w' = \frac{v'}{\rho} - \frac{v}{\rho^2}$

$\therefore h^2 \left(\frac{v''}{\rho} - \cancel{\frac{2v'}{\rho^2}} + \cancel{\frac{2v}{\rho^3}} + \frac{2}{\rho}\left(\cancel{\frac{v'}{\rho}} - \cancel{\frac{v}{\rho^2}} \right) \right) = c \frac{v}{\rho}$

$\therefore h^2 v'' = c v$

$\therefore (h^2 D^2 - c) v = 0 \quad \therefore v = \begin{cases} B_1 \sin k\rho + B_2 \cos k\rho & c<0, \text{ where } k = \sqrt{\frac{-c}{h^2}} \\ B_3 + B_4 \rho & c=0 \\ B_5 e^{+k'\rho} + B_6 e^{-k'\rho} & c>0, \text{ where } k' = \sqrt{\frac{c}{h^2}} \end{cases}$

$\therefore u = g w = g \frac{v}{\rho} = A e^{ct} \cdot \frac{v}{\rho}$ where v is one of the above functions

But $u(t, R) = 0 \quad \therefore A e^{ct} \cdot \frac{v(R)}{R} = 0 \quad \therefore v(R) = 0$

It can be seen that $u(t, \rho) = u(t, -\rho)$ (ρ-Symmetric)

$\therefore v(R) = v(-R) = 0$

$\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{cases} \begin{bmatrix} B_1 \sin kR + B_2 \cos kR \\ -B_1 \sin kR + B_2 \cos kR \end{bmatrix} & c<0 \quad \begin{cases} \Delta = 2 \sin kR \cos kR = \sin 2kR \neq 0 \therefore B_1 = B_2 = 0 \therefore X \\ \text{OR } \Delta = 0 \therefore 2kR = n\pi \begin{cases} n \text{ odd} \therefore kR = \frac{\pi}{2}, \frac{3\pi}{2}, \dots; \therefore B_1 = 0 \\ n \text{ even} \therefore kR = 0\pi, 2\pi, \dots; \quad B_2 = 0 \end{cases} \end{cases} \\ \begin{bmatrix} B_3 + B_4 R \\ B_3 - B_4 R \end{bmatrix} \Longrightarrow B_3 = B_4 = 0 \quad \therefore X \ (\text{because } \Delta = -2R \neq 0) & c=0 \\ \begin{bmatrix} B_5 e^{k'R} + B_6 e^{-k'R} \\ B_5 e^{-k'R} + B_6 e^{+k'R} \end{bmatrix} \Rightarrow B_5 = B_6 = 0 \quad \therefore X \ (\text{because } \Delta = e^{2k'R} - e^{-2k'R} \neq 0) & c>0 \end{cases}$

$\therefore$ To have a solution c must be negative and $2kR = (\text{must})\, n\pi$

$\therefore k = \frac{n\pi}{2R} \quad \therefore c = -k^2 h^2 = -\left(\frac{n\pi}{2R}\right)^2 \cdot h^2 = -\left(\frac{n\pi h}{2R}\right)^2, \quad n = 0, 1, 2, \dots$

$\therefore u(t, \rho) = A e^{ct} \cdot \frac{v}{\rho} = \frac{A e^{-\left(\frac{n\pi h}{2R}\right)^2 t}}{\rho} \cdot \left(B_1 \sin \frac{n\pi\rho}{2R} + B_2 \cos \frac{n\pi\rho}{2R} \right) \begin{cases} n \text{ odd} & B_1 = 0 \\ n \text{ even} & B_2 = 0 \end{cases}$

$= \begin{cases} n \text{ odd} & \frac{B_2'}{\rho} e^{-\left(\frac{n\pi h}{2R}\right)^2 t} \cdot \cos \frac{n\pi\rho}{2R} \\ n \text{ even} & \frac{B_1'}{\rho} e^{-\left(\frac{n\pi h}{2R}\right)^2 t} \cdot \sin \frac{n\pi\rho}{2R} \end{cases}$ but $u(t, \rho) = u(t, -\rho)$

$\therefore \begin{array}{l} \text{odd } n \\ \text{even } n \end{array} \begin{bmatrix} \frac{B_2'}{\rho} e^{-\left(\frac{n\pi h}{2R}\right)^2 t} \cdot \cos \frac{n\pi\rho}{2R} \\ \frac{B_1'}{\rho} e^{-\left(\frac{n\pi h}{2R}\right)^2 t} \cdot \sin \frac{n\pi\rho}{2R} \end{bmatrix} = \begin{bmatrix} -\frac{B_2'}{\rho} e^{-\left(\frac{n\pi h}{2R}\right)^2 t} \cdot \cos \frac{n\pi\rho}{2R} \\ +\frac{B_1'}{\rho} e^{-\left(\frac{n\pi h}{2R}\right)^2 t} \cdot \sin \frac{n\pi\rho}{2R} \end{bmatrix} \begin{array}{l} \Rightarrow \therefore B_2' = -B_2' \quad \therefore B_2' = 0 \\ \text{OK} \end{array}$

$\therefore\ u(t,\rho) = \dfrac{B_1'}{\rho} \cdot e^{-\left(\frac{n\pi h}{2R}\right)^2 t} \cdot \sin\dfrac{n\pi\rho}{2R}$ (n even) $= \dfrac{B_m}{\rho} \cdot e^{-\left(\frac{m\pi h}{R}\right)^2 t} \cdot \sin\dfrac{m\pi\rho}{R}$, $m = 1, 2, 3, \ldots$

$$\therefore\ u(t,\rho) = \sum_{m=1}^{\infty} \frac{B_m}{\rho} \cdot e^{-\left(\frac{m\pi h}{R}\right)^2 t} \cdot \sin\frac{m\pi\rho}{R}$$

But $u(0,\rho) = f(\rho)$ $\quad\therefore\ f(\rho) = u(0,\rho) = \displaystyle\sum_{m=1}^{\infty} \frac{B_m}{\rho} \sin\frac{m\pi\rho}{R}$

$= \dfrac{1}{\rho} \cdot \displaystyle\sum_{m=1}^{\infty} B_m \sin\frac{m\pi\rho}{R}$ $\quad\therefore\ \rho f(\rho) = \displaystyle\sum_{m=1}^{\infty} B_m \sin\frac{m\pi\rho}{R}$

By odd extension of Fourier Series (sine coefficient $= \dfrac{4}{P}\displaystyle\int_0^{P/2} \rho f(\rho) \sin\frac{m\pi\rho}{R}\, d\rho$, P = period at $m=1$)

$\therefore\ B_m = \dfrac{4}{2R}\displaystyle\int_0^R \rho f(\rho) \sin\frac{m\pi\rho}{R}\, d\rho = \frac{2}{R}\int_0^R \rho f(\rho) \sin\frac{m\pi\rho}{R}\, d\rho.$

$\therefore$ The Solution is $u(t,\rho) = \dfrac{1}{\rho}\displaystyle\sum_{m=1}^{\infty} B_m \cdot e^{-\left(\frac{m\pi h}{R}\right)^2 t} \cdot \sin\frac{m\pi\rho}{R}$

where $B_m = \dfrac{2}{R}\displaystyle\int_0^R \rho f(\rho) \sin\frac{m\pi\rho}{R}\, d\rho.$

$\frac{2}{480}$ $u(t, \rho)$ is required such that:

$u = u_0 \; (= f(\rho))$ at $t = 0$, $\rho \in [0, R]$ $\therefore u(0,\rho) = u_0$

$u = u_1 =$ constant at $t > 0$ & $\rho = R$ $\therefore u(t, R) = u_1$ $\quad \therefore u_t(t,R) = 0$

$\therefore u_t = h^2 \left(u_{\rho\rho} + \frac{2}{\rho} u_\rho\right)$ proceed just as in the above problem $\left(\frac{1}{480}\right)$

$$\therefore u(t,\rho) = \begin{cases} c<0 & e^{ct} \cdot \dfrac{B_1 \sin k\rho + B_2 \cos k\rho}{\rho} \qquad k = \sqrt{\frac{-c}{h^2}} \\ c=0 & e^{ct} \cdot \dfrac{B_3 + B_4 \rho}{\rho} \\ c>0 & e^{ct} \cdot \dfrac{B_5 e^{k'\rho} + B_6 e^{-k'\rho}}{\rho} \qquad k' = \sqrt{\frac{c}{h^2}} \end{cases}$$

But $u_t(t, R) = 0 = u(t, -R)$

$$\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{cases} c<0 & \dfrac{ce^{ct}}{R} \cdot \begin{bmatrix} B_1 \sin kR + B_2 \cos kR \\ B_1 \sin kR - B_2 \cos kR \end{bmatrix} \begin{cases} \Delta = -2\sin kR \cos kR = -\sin 2kR \neq 0 & \therefore B_1 = B_2 = 0, \therefore \times \\ \Delta = 0 \; \therefore 2kR = n\pi & \begin{cases} n \text{ odd} & kR = \frac{\pi}{2}, \frac{3\pi}{2}, \dots ; \; B_1 = 0 \\ n \text{ even} & kR = 0, \pi, 2\pi, \dots ; \; B_2 = 0 \end{cases} \end{cases} \\ c=0 & \dfrac{ce^{ct}}{R} \cdot \begin{bmatrix} B_3 + B_4 R \\ -B_3 + B_4 R \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ because } c = 0, \text{ OK but } \begin{bmatrix} u_1 \\ u_1 \end{bmatrix} = \frac{1}{R}\begin{bmatrix} B_3 + B_4 R \\ -B_3 + B_4 R \end{bmatrix}, \Delta \neq 0 \therefore B_3 = 0 \;\&\; B_4 = u_1 \\ c>0 & \dfrac{ce^{ct}}{R} \cdot \begin{bmatrix} B_5 e^{k'R} + B_6 e^{-k'R} \\ -B_5 e^{k'R} - B_6 e^{+k'R} \end{bmatrix} \; \Delta = -e^{2k'R} + e^{-2k'R} \neq 0 \therefore B_5 \;\&\; B_6 \text{ are both zero}, \therefore \times \end{cases}$$

$$\therefore u(t,\rho) = \begin{cases} c=0 & u_1 \quad \text{OK} \\ c<0 \; (2kR = n\pi) & \begin{cases} \text{odd } n & B_2 e^{ct} \dfrac{\cos k\rho}{\rho} \quad \text{but } u(t,\rho) = u(t,-\rho) \therefore B_2 = -B_2 \therefore B_2 = 0 \\ \text{even } n & B_1 e^{ct} \dfrac{\sin k\rho}{\rho} \quad \text{OK } (u(t,\rho) = u(t,-\rho)) \end{cases} \end{cases}$$

$\therefore$ To have a varying solution C must be negative and $2kR = n\pi$ where n is even

$\therefore k = \frac{n\pi}{2R}$ (n even) $= \frac{m\pi}{R}$, $m = 0, 1, 2, \dots$ $\quad \therefore c = -k^2h^2 = -\left(\frac{m\pi h}{R}\right)^2$

$$\therefore u(t,\rho) = u_1 + B_1 \frac{e^{-\left(\frac{m\pi h}{R}\right)^2 t}}{\rho} \cdot \sin\frac{m\pi\rho}{R}, \; m = 0, 1, 2, \dots = u_1 + \sum_{m=1}^{\infty} B_m \frac{e^{-\left(\frac{m\pi h}{R}\right)^2 t}}{\rho} \cdot \sin\frac{m\pi\rho}{R}$$

but $u(0,\rho) = u_0$ $\quad \therefore u_0 = u(0,\rho) = u_1 + \frac{1}{\rho}\sum_{m=1}^{\infty} B_m \sin\frac{m\pi\rho}{R}$

$\therefore (u_0 - u_1)\rho = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi\rho}{R}$

$\therefore$ By odd extension of Fourier Series $\therefore B_m = \frac{4}{2R}\int_0^R (u_0-u_1)\rho \sin\left(\frac{m\pi\rho}{R}\right) d\rho =$

$$= \frac{2(u_0-u_1)}{R}\left[-\rho\frac{\cos(m\pi\rho/R)}{(m\pi/R)} + \cancel{\frac{\sin(m\pi\rho/R)}{(m\pi/R)^2}}^{\,0}\right]_0^R = -\frac{2(u_0-u_1)}{m\pi}(R\cos m\pi - 0) = \frac{2(u_1-u_0)}{m\pi} R \cdot (-1)^m$$

$$\therefore u(t,\rho) = u_1 + \frac{2(u_1-u_0)R}{\rho\pi} \cdot \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cdot e^{-\left(\frac{m\pi h}{R}\right)^2 t} \cdot \sin\frac{m\pi\rho}{R}.$$

$\boxed{\frac{2}{482}}$

$y_t(0,x) = 0 \quad \text{all } x \quad (1)$

$y(0,x) = \frac{x(c-x)}{c} = f(x) \qquad x \in (0,c) \quad (2)$

$y(t,0) = y(t,c) = 0 \quad (3)$

$\therefore\ y_{tt} = a^2 y_{xx}$, Let $y(t,x) = g(t)\cdot w(x)$

$\therefore\ g''w = a^2 g w'' \Rightarrow \frac{g''}{g} = \frac{a^2 w''}{w} = b$

$\therefore\ g(t) = \begin{cases} A_1 t + A_2 & b=0 \\ A_3 e^{kt} + A_4 e^{-kt} & 0<b=k^2 \\ A_5 \sin kt + A_6 \cos kt & 0>b=-k^2 \end{cases}$

& $w(x) = \begin{cases} B_1 x + B_2 & b=0 \\ B_3 e^{kx/a} + B_4 e^{-kx/a} & 0<b=k^2 \\ B_5 \sin(kx/a) + B_6 \cos(kx/a) & 0>b=-k^2 \end{cases}$

for $b=0$

from (3) $\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} = g(t)\begin{bmatrix} w(0) \\ w(c) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & 1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \therefore \Delta = -c \neq 0 \quad \therefore B_1 = B_2 = 0$

for $b>0$

$\therefore$ from (3) $\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} = g(t)\begin{bmatrix} w(0) \\ w(c) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{kc/a} & e^{-kc/a} \end{bmatrix}\begin{bmatrix} B_3 \\ B_4 \end{bmatrix}$

$\therefore\ \Delta = e^{-kc/a} - e^{kc/a} = -2\sinh(kc/a) \quad \because k \neq 0$ & $c \neq 0 \therefore \Delta \neq 0$

$\therefore\ B_3 = B_4 = 0$

for $b<0$

$\therefore$ from (3) $\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} = g(t)\begin{bmatrix} w(0) \\ w(c) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin(\frac{kc}{a}) & \cos(\frac{kc}{a}) \end{bmatrix}\begin{bmatrix} B_5 \\ B_6 \end{bmatrix}$

$\therefore\ \Delta = -\sin(\frac{kc}{a}) \quad \therefore$ either $\Delta \neq 0 \quad \therefore B_5 = B_6 = 0$

or $\Delta = 0 \quad \therefore \frac{kc}{a} = n\pi \quad \therefore k = \frac{n\pi a}{c}$, n integer

& hence $B_6 = 0$ & B_5 is arbitrary

$\therefore$ Solution is only there when $b < 0$

$\therefore$ $y(t, x) = g(t) \cdot w(x) = \sum_{\text{all } k} (A_5 \sin kt + A_6 \cos kt) B_5 \sin \frac{kx}{a}$

$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{c}\right) \cdot \left[A_n \sin\left(\frac{n\pi at}{c}\right) + B_n \cos\left(\frac{n\pi at}{c}\right)\right]$

Applying (1)

$\therefore$ $y_t(0, x) = 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{c}\right)\left[A_n \cdot \left(\frac{n\pi a}{c}\right) + 0\right]$ $\therefore A_n = 0$ for all n

$\therefore$ $y(t, x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{c}\right) \cos\left(\frac{n\pi at}{c}\right)$

Applying (2) $\therefore$ $y(0, x) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{c}\right)$

$\therefore$ $B_n = \frac{2}{c} \int_0^c f(x) \cdot \sin \frac{n\pi x}{c}\, dx = \frac{2}{c} \int_0^c \frac{x(c-x)}{c} \sin\left(\frac{n\pi x}{c}\right) dx =$

$= \frac{2}{c^2} \int_0^c (cx - x^2)\, d\, \frac{\cos(n\pi x/c)}{-(n\pi/c)} =$

$= \frac{2}{-n\pi c} \cdot \left[(cx - x^2) \cos(n\pi x/c)\Big|_0^c - \int_0^c \cos(n\pi x/c) \cdot (c - 2x)\, dx\right] =$

$= \frac{2}{-n\pi c}\left[0 - 0 - \int_0^c (c - 2x)\, d\, \frac{\sin(n\pi x/c)}{(n\pi/c)}\right] =$

$= \frac{2}{(n\pi)^2} \cdot \left[(c - 2x) \sin(n\pi x/c)\Big|_0^c - \int_0^c \sin(n\pi x/c)(-2)\, dx\right] =$

$= \frac{2}{n^2\pi^2} \cdot \left[-c \overset{0}{\cancel{\sin(n\pi)}} - 0 + 2 \cdot \frac{\cos(n\pi x/c)}{-n\pi/c}\Big|_0^c\right] = -\frac{4c}{n^3\pi^3}\left(\cos(n\pi) - 1\right)$

$= 0$ for even n & $\left(\frac{2}{n\pi}\right)^3 c$ for odd n

$\therefore$ $y(t, x) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \left(\frac{2}{n\pi}\right)^3 c \cdot \sin\left(\frac{n\pi x}{c}\right) \cdot \cos\left(\frac{n\pi at}{c}\right)$ put $n = 2m+1$

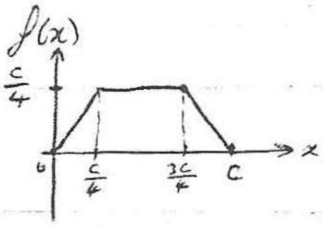
$$= \frac{8c}{\pi^3} \cdot \sum_{m=0}^{\infty} \frac{\sin[(2m+1)\pi x/c] \cdot \cos[(2m+1)\pi at/c]}{(2m+1)^3}$$

$\boxed{\frac{3}{482}}$

$$y_{tt} = a^2 y_{xx}$$

$$y(t,0) = y(t,c) = 0$$

$$y(0,x) = f(x) = \begin{cases} x & x \in [0, c/4] \\ \frac{c}{4} & x \in (\frac{c}{4}, 3c/4] \\ c - x & x \in (\frac{3c}{4}, c] \end{cases}$$



$$y_t(0,x)$$

Let $y = w(t) \cdot g(x)$ $\quad \therefore y(t,0) = y(t,c) = 0 \Rightarrow g(0) = g(c) = 0$

& $g w'' = a^2 g'' w$ $\quad \therefore \frac{w''}{w} = \frac{a^2 g''}{g} = k$

$\therefore a^2 g'' - kg = 0$

$\therefore (a^2 D^2 - k) g = 0$

$k<0$: $g(x) = A_1 \sin \alpha x + A_2 \cos \alpha x$ where $\alpha = \sqrt{-k}/a$

$k=0$: $g(x) = A_3 + A_4 x$

$k>0$: $g(x) = A_5 e^{\beta x} + A_6 e^{-\beta x}$ where $\beta = \sqrt{k}/a$

but $\begin{bmatrix} g(0) \\ g(c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\therefore$

$k<0$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_2 \\ A_1 \sin \alpha c + A_2 \cos \alpha c \end{bmatrix}$, $\Delta = -\sin \alpha c$ $<$ $\therefore A_1 = A_2 = 0$ $\therefore X$; $\Delta = 0$ $\therefore \alpha c = m\pi$ & $A_2 = 0$, OK

$k=0$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_3 \\ A_3 + A_4 c \end{bmatrix}$ $\quad \Delta = c \neq 0$ $\therefore A_3 = A_4 = 0$ $\therefore$

$k>0$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_5 + A_6 \\ A_5 e^{\beta c} + A_6 e^{-\beta c} \end{bmatrix}$ $\quad \Delta = e^{-\beta c} - e^{\beta c} \neq 0$ $\therefore A_5 = A_6 = 0$ $\therefore X$

$\therefore$ To have a solution k must be negative & $\alpha = \frac{m\pi}{c}$ & $k = -\alpha^2 a^2 = -\left(\frac{m\pi a}{c}\right)^2$

$\therefore g(x) = A_1 \sin \alpha x = A_1 \sin \frac{m\pi}{c} \cdot x$ $\quad , m = 1, 2, 3, \ldots$

& $\frac{w''}{w} = k = -\left(\frac{m\pi a}{c}\right)^2$ $\quad \therefore \left[D^2 + \left(\frac{m\pi a}{c}\right)^2\right] w = 0$ $\quad \therefore w = B_1 \sin \frac{m\pi a t}{c} + B_2 \cos \frac{m\pi a t}{c}$

But $y_t(0,x) = 0$ $\quad \therefore w'(0) \cdot g(x) = 0$ $\quad \therefore w'(0) = 0$ $\quad \therefore \frac{m\pi a}{c}\left[B_1 \cos \frac{m\pi a t}{c} - B_2 \sin \frac{m\pi a t}{c}\right]_{t=0}$

$= \frac{m\pi a}{c}[B_1 - 0] = 0$ $\quad \therefore B_1 = 0$

$\therefore w(t) = B_2 \cos \frac{m\pi a t}{c}$

$\therefore y(t,x) = w(t) \cdot g(x) = B_2 \cos \frac{m\pi a t}{c} \cdot A_1 \sin \frac{m\pi x}{c}$ $\quad , m = 1, 2, 3, \ldots$

$\therefore y(t,x) = \sum_{m=1}^{\infty} C_m \cos \frac{m\pi a t}{c} \cdot \sin \frac{m\pi x}{c}$

But $y(0,x) = f(x)$ $\quad \therefore f(x) = \sum_{m=1}^{\infty} C_m \sin \frac{m\pi x}{c}$ $\quad$, Period $= 2c$ (at $m=1$)

$\therefore$ By odd Fourier Extension $\therefore C_m = \frac{2}{c} \int_0^c f(x) \cdot \sin \frac{m\pi x}{c}\, dx = $ (valued at odd m only, zero otherwise)

$= \frac{4}{c} \int_0^{c/2} f(x) \sin \frac{m\pi x}{c}\, dx = \frac{4}{c}\left[\int_0^{c/4} x \sin\left(\frac{m\pi x}{c}\right) dx + \int_{c/4}^{c/2} \frac{c}{4} \sin \frac{m\pi x}{c}\, dx\right] = \frac{4}{c}\left[\frac{-cx}{m\pi} \cos\left(\frac{m\pi x}{c}\right)\right.$

$\left. + \frac{c^2}{m^2 \pi^2} \cdot \sin \frac{m\pi x}{c} \Big|_0^{c/4} - \frac{c^2}{4m\pi} \cos \frac{m\pi x}{c}\right]_{c/4}^{c/2} = \frac{4}{c}\left[\cancel{\frac{-c^2}{4m\pi} \cos \frac{m\pi}{4}} - 0 + \frac{c^2}{m^2\pi^2} \sin \frac{m\pi}{4} - 0 - \frac{c^2}{4m\pi} \cos \frac{m\pi}{2}\right.$

$\left. + \cancel{\frac{c^2}{4m\pi} \cdot \cos \frac{m\pi}{4}}\right] = \frac{2c}{\pi^2}\left[\frac{2}{m^2} \sin\left(\frac{m\pi}{4}\right) - \frac{\pi}{2m} \cancel{\cos \frac{m\pi}{2}}_{0}\right] = \frac{4c}{m^2\pi^2} \sin \frac{m\pi}{4}$ $\quad m = 1, 3, 5, \ldots$ odd

$\therefore \quad y(t,x) = \frac{4C}{\pi^2} \cdot \sum_{m=1}^{\infty} \frac{1}{m^2} \cdot \sin\frac{m\pi}{4} \cdot \cos\frac{m\pi a t}{c} \sin\frac{m\pi x}{c}$ m odd

$= \frac{4C}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cdot \sin\left(\frac{2n-1}{4} \cdot \pi\right) \cdot \cos\frac{(2n-1)\pi a t}{c} \cdot \sin\frac{(2n-1)\pi x}{c}$ all n

Note:	value of $\frac{2}{m^2} \sin\frac{m\pi}{4}$	value of $\frac{1}{m^2}\left(\sin\frac{m\pi}{4} + \sin\frac{3m\pi}{4}\right)$
$m=1$	$\frac{2}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$
$m=2$	not applicable	0
$m=3$	$\frac{2}{9\sqrt{2}}$	$\frac{2}{9\sqrt{2}}$
$m=4, \ldots$ even	not applicable	0
$m=5$	$-\frac{2}{25\sqrt{2}}$	$-\frac{2}{25\sqrt{2}}$
$m=7$	$-\frac{2}{49\sqrt{2}}$	$-\frac{2}{49\sqrt{2}}$

$\therefore$ expressions are equivalent.

$\boxed{\frac{6}{483}}$

$$y_{tt} = a^2 y_{xx}\ ,\quad y(0,x) = 0\ ,\quad y_t(0,x) = \phi(x) = \frac{ax(c-x)}{4c^2}$$

$$\& \quad y(t,0) = y(t,c) = 0 \qquad x \in [0,c]$$

Let $y = g(t)\cdot f(x)$

$$g'' f = a^2 f'' g \qquad \therefore \frac{g''}{4} = \frac{a^2 f''}{6} \quad c'$$

$$g'' = c' g \ \therefore (D^2 - c')g = 0$$

$$\therefore g(t) = \begin{cases} A_1 \sin kt + A_2 \cos kt & c'<0 \quad \text{where } k = \sqrt{-c'} \\ A_3 + A_4 t & c'=0 \\ A_5 e^{bt} + A_6 e^{-bt} & c'>0 \quad \text{where } b = \sqrt{c'} \end{cases}$$

$$\& \quad a^2 f'' = c' f \ \therefore (a^2 D^2 - c') f = 0$$

$$\therefore f(x) = \begin{cases} B_1 \sin \frac{kx}{a} + B_2 \cos \frac{kx}{a} & c'<0 \\ B_3 + B_4 x & c'=0 \\ B_5 e^{\frac{bx}{a}} + B_6 e^{-\frac{bx}{a}} & c'>0 \end{cases}$$

$$\therefore y = g(t) f(x)\ .\ \text{but } y(t,0) = y(t,c) = 0 \ \therefore g(t)\cdot f(0) = g(t)\cdot f(c) = 0 \quad \text{or} \begin{bmatrix} f(0) \\ f(c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{cases} c'<0 & \begin{bmatrix} 0 + B_2 \\ B_1 \sin \frac{kc}{a} + B_2 \cos \frac{kc}{a} \end{bmatrix} \ \Delta = -\sin \frac{kc}{a} \begin{cases} \neq 0 & \therefore B_1 = B_2 = 0 \ \therefore X \\ = 0 & \therefore \frac{kc}{a} = n\pi\ ,\ n = 1,2,3,\dots \ \& \ B_2 = 0\ ,\ \text{OK} \end{cases} \\ c'=0 & \begin{bmatrix} B_3 \\ B_3 + cB_4 \end{bmatrix} \ \Delta = c \neq 0 \ \therefore B_3 = B_4 = 0 \ \therefore X \\ c'>0 & \begin{bmatrix} B_5 + B_6 \\ B_5 e^{\frac{bc}{a}} + B_6 e^{-\frac{bc}{a}} \end{bmatrix} \ \Delta = e^{-\frac{bc}{a}} - e^{\frac{bc}{a}} \neq 0 \ \therefore B_5 = B_6 = 0 \ \therefore X \end{cases}$$

$\therefore$ To have a solution c' is negative & $k = \frac{n\pi a}{c}$ $\therefore c' = -k^2 = -\left(\frac{n\pi a}{c}\right)^2$

$$\therefore y(t,x) = (A_1 \sin kt + A_2 \cos kt)\cdot B_1 \sin \frac{kx}{a} = \left(A_1' \sin \frac{n\pi a t}{c} + A_2' \cos \frac{n\pi a t}{a}\right)\cdot \sin \frac{n\pi x}{c}$$

But $y(0,x) = 0$ $\therefore 0 = y(0,x) = A_2' \sin \frac{n\pi x}{c}$ $\therefore A_2' = 0$

$$\therefore y(t,x) = A_1' \sin \frac{n\pi a t}{c} \sin \frac{n\pi x}{c} \ (;\ n = 1,2,3,\dots) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi a t}{c} \sin \frac{n\pi x}{c}$$

But $y_t(0,x) = \phi(x)$ $\therefore \phi(x) = y_t(0,x) = \sum_{n=1}^{\infty} A_n \frac{n\pi a}{c} \cos \frac{n\pi a t}{c}\Big|_{t=0} \cdot \sin \frac{n\pi x}{c} = \frac{\pi a}{c} \sum_{n=1}^{\infty} n A_n \sin \frac{n\pi x}{c}$

$\therefore \frac{c\phi(x)}{\pi a} = \sum_{n=1}^{\infty} n A_n \sin \frac{n\pi x}{c}$. Using odd extension of Fourier Series:

$$\therefore n A_n = \frac{2}{c} \int_0^c \frac{c\phi(x)}{\pi a} \sin \frac{n\pi x}{c} dx = \frac{2}{\pi a} \int_0^c \frac{ax(c-x)}{4c^2} \sin \frac{n\pi x}{c} dx = \frac{1}{2\pi c^2} \left[\frac{+x^2 \cos(n\pi x/c)}{(n\pi/c)} - \frac{2x \sin(n\pi x/c)}{(n\pi/c)^2} \right.$$

$$\left. - \frac{2\cos(n\pi x/c)}{(n\pi/c)^3} - \frac{cx \cos(n\pi x/c)}{(n\pi/c)} + \frac{c \sin(n\pi x/c)}{(n\pi/c)^2} \right]_0^c = \frac{1}{2\pi c^2} \left[\frac{c^3}{n\pi}(-1)^n - \frac{2c^3}{n^3\pi^3}\left((-1)^n - 1\right) - \frac{c^3}{n\pi}(-1)^n \right] = \begin{cases} 0 & n = 2,4,\dots \\ \frac{2c}{n^3\pi^4} & \text{odd} \end{cases}$$

$$\therefore A_n = \frac{2c}{n^4\pi^4} \ (n\ \text{odd}) \ \therefore y(t,x) = \sum_{m=1}^{\infty} \frac{2c}{\pi^4 (2m-1)^4} \cdot \sin\left[\frac{(2m-1)\pi a t}{c}\right] \sin\left[\frac{(2m-1)\pi x}{c}\right].$$

$\boxed{\frac{7}{483}}$

$$y_{tt} = a^2 y_{xx}$$

$y(t, 0) = y(t, c) = 0$ (1)

$y(0, x) = 0$ (2)

$y_t(0, x) = \phi(x) = \begin{cases} 0 & x \in (0, \frac{c}{3}) \\ V_0 & x \in (\frac{c}{3}, \frac{2c}{3}) \\ 0 & x \in (\frac{2c}{3}, c) \end{cases}$ (3)

$\therefore y(t, x) = f(x) \cdot g(t)$

$\therefore g'' f = a^2 f'' g \Rightarrow \frac{g''}{g} = \frac{a^2 f''}{f} = b$

for $b = 0$

$\therefore g(t) = A_1 t + A_2$ & $f(x) = B_1 x + B_2$ $\therefore$ from (1)

$\therefore \begin{pmatrix} 0 \\ 0 \end{pmatrix} = g(t) \begin{pmatrix} f(0) \\ f(c) \end{pmatrix} = g(t) \begin{pmatrix} 0 & 1 \\ c & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ $\because g(t) \neq 0$ & $c \neq 0$ $\therefore \Delta \neq 0$ $\therefore B_1 = B_2 = 0$, nothing.

for ~~c~~ $b = k^2$

$\therefore g(t) = A_3 e^{kt} + A_4 e^{-kt}$ & $f(x) = B_3 e^{kx/a} + B_4 e^{-kx/a}$ $\therefore$ from (1)

$\therefore \begin{pmatrix} 0 \\ 0 \end{pmatrix} = g(t) \cdot \begin{pmatrix} f(0) \\ f(c) \end{pmatrix} = g(t) \begin{pmatrix} 1 & 1 \\ e^{ck/a} & e^{-ck/a} \end{pmatrix} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}$ $\because g(t) \neq 0$ & $k \neq 0$ $\therefore \Delta = -2 \sinh \frac{ck}{a} \neq 0$ $\therefore B_3 = B_4 = 0$, nothing.

for ~~c~~ $b = -k^2$,

$\therefore g(t) = A_5 \sin kt + A_6 \cos kt$ & $f(x) = B_5 \sin(kx/a) + B_6 \cos(kx/a)$

$\therefore$ from (1): $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = g(t) \begin{pmatrix} f(0) \\ f(c) \end{pmatrix} = g(t) \begin{pmatrix} 0 & 1 \\ \sin ck/a & \cos(ck/a) \end{pmatrix} \begin{pmatrix} B_5 \\ B_6 \end{pmatrix}$ $\because g(t) \neq 0$ & $k \neq 0$ $\therefore \Delta = -\sin(ck/a) \neq 0$

$\Rightarrow B_5 = B_6 = 0$, or $\Delta = -\sin(ck/a) = 0 \Rightarrow \frac{ck}{a} = n\pi$ & $B_6 = 0$, n integer

$\therefore k = \frac{an\pi}{c}$ & $y(t, x) = \sum_k (A_5 \sin kt + A_6 \cos kt) \cdot B_5 \sin(kx/a)$

$\therefore y(t,x) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{an\pi t}{c}\right) + B_n \cos\left(\frac{an\pi t}{c}\right) \right] \sin\left(\frac{n\pi x}{c}\right)$

from (2)

$\because y(0, x) = 0 \therefore \sum_{n=1}^{\infty} B_n \sin(n\pi x/c) = 0$ $\therefore B_n = 0$ for all n

$\therefore y(t, x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{an\pi t}{c}\right) \sin\left(\frac{n\pi x}{c}\right)$ & from (3)

$\therefore y_t(0, x) = \phi(x) = \sum_{n=1}^{\infty} A_n \cdot \left(\frac{an\pi}{c}\right) \sin \frac{n\pi x}{c}$

$\therefore A_n \cdot \left(\frac{an\pi}{c}\right) = \frac{2}{c} \int_0^c \phi(x) \sin \frac{n\pi x}{c} dx = \frac{2}{c} \int_{c/3}^{2c/3} V_0 \sin \frac{n\pi x}{c} dx = \frac{2V_0}{c} \cdot \frac{\cos(n\pi x/c)}{-n\pi/c} \Big|_{c/3}^{2c/3} =$

$= \frac{-2V_0}{n\pi} \left(\cos(2n\pi/3) - \cos(n\pi/3) \right) \Rightarrow A_n = \frac{2cV_0}{an^2\pi^2} \left(\cos(n\pi/3) - \cos(2n\pi/3) \right)$

$\therefore y(t, x) = \frac{2cV_0}{a\pi^2} \cdot \sum_{n=1}^{\infty} \left[\frac{\cos(n\pi/3) - \cos(2n\pi/3)}{n^2} \right] \sin\left(\frac{n\pi at}{c}\right) \sin\left(\frac{n\pi x}{c}\right)$

$\frac{4}{486}$

$u_{xx} + u_{yy} = 0$, $u(0,y) = u(a,y) = 0$, $u_y(x,0) = 0$, $u(x,b) = f(x)$

Let $u = g(x) \cdot w(y)$

$\therefore g'' w + g w'' = 0 \quad \therefore \frac{g''}{g} + \frac{w''}{w} = 0 \quad \therefore \frac{g''}{g} = \frac{-w''}{w} = c$

$$\therefore u(\text{straight}) = \begin{cases} c<0 & (-c = \alpha^2) \quad (A_1 \sin\alpha x + A_2 \cos\alpha x)\cdot(B_1 e^{\alpha y} + B_2 e^{-\alpha y}) \\ c=0 & (A_3 + A_4 x)(B_3 + B_4 y) \\ c>0 & (c = \beta^2) \quad (A_5 e^{\beta x} + A_6 e^{-\beta x})\cdot(B_5 \sin\beta y + B_6 \cos\beta y) \end{cases}$$

$$\text{but } \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}u(0,y)\\u(a,y)\end{bmatrix} \quad \therefore \begin{bmatrix}0\\0\end{bmatrix} = \begin{cases} c<0 & \begin{bmatrix}0 + A_2 \\ A_1 \sin\alpha a + A_2 \cos\alpha a\end{bmatrix} \quad \Delta = -\sin\alpha a \begin{cases} \neq 0 & \therefore A_1 = A_2 = 0 \therefore X \\ = 0 & \therefore \alpha a = n\pi,\ A_2 = 0, OK \end{cases} \\ c=0 & \begin{bmatrix}A_3 \\ A_3 + A_4 a\end{bmatrix} \quad \Delta \neq 0 \quad \therefore A_3 = A_4 = 0 \quad \therefore X \\ c>0 & \begin{bmatrix}A_5 + A_6 \\ A_5 e^{\beta a} + A_6 e^{-\beta a}\end{bmatrix} \quad \Delta = e^{-\beta a} - e^{\beta a} \neq 0 \quad \therefore A_5 = A_6 = 0, \therefore X \end{cases}$$

$\therefore$ To have a solution c is negative & $\alpha a = n\pi$, $A_2 = 0$

but $c = -\alpha^2$ & $\alpha = \frac{n\pi}{a} \quad \therefore c = -\left(\frac{n\pi}{a}\right)^2$

$\therefore u(x,y) = A_1 \sin\alpha x \cdot (B_1 e^{\alpha y} + B_2 e^{-\alpha y}) = \sin\frac{n\pi x}{a}\left(B_1' e^{\alpha y} + B_2' e^{-\alpha y}\right)$

But $u_y(x,0) = 0 \quad \therefore \alpha\left(B_1' e^{\alpha y} - B_2' e^{-\alpha y}\right)\Big|_{y=0} = \alpha(B_1' - B_2') = 0 \quad \therefore B_1' = B_2'$

$\therefore u(x,y) = B_1' \sin\frac{n\pi x}{a}\left(e^{\alpha y} + e^{-\alpha y}\right) = B'' \sin\frac{n\pi x}{a}\cdot\cosh\frac{n\pi y}{a}$, $n = 1,2,3$

$\therefore u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \cosh\frac{n\pi y}{a}$

But, $u(x,b) = f(x)$

$\therefore f(x) = u(x,b) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \cosh\frac{n\pi b}{a} = \sum_{n=1}^{\infty}\left(B_n \cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$

Using odd extension of Fourier Series:

$\therefore B_n \cosh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\cdot\sin\frac{n\pi x}{a}\,dx \quad \therefore B_n = \frac{2}{a\cosh(\frac{n\pi b}{a})}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$

$\therefore u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\cosh\frac{n\pi y}{b}$ with B_n as found above.

Assuming $f(x) = u_0 \quad \therefore B_n = \frac{2}{a\cosh(n\pi b/a)}\cdot u_0\left[\frac{-\cos(n\pi x/a)}{n\pi/a}\right]_0^a = \frac{-2u_0[(-1)^n - 1]}{n\pi\cosh(n\pi b/a)}$

$$= \begin{cases} n \text{ odd} & \frac{4u_0}{n\pi\cosh(n\pi b/a)} \\ n \text{ even} & 0 \end{cases}$$

$$\therefore u(x,y) = \sum_{m=1}^{\infty} \frac{4u_0 \sin\left[\frac{(2m-1)\pi x}{a}\right]\cosh\left[\frac{(2m-1)\pi y}{b}\right]}{(2m-1)\pi\cosh[(2m-1)b\pi/a]}.$$

$$\therefore u\left(\frac{a}{2},0\right) = \frac{4u_0}{\pi}\sum_{m=0}^{\infty} \sin[(2m+1)\pi/2] / \{(2m+1)\cosh[(2m+1)\pi b/a]\}$$

$$= \frac{4u_0}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)\cosh[(2m+1)\pi b/a]} = 97$$

$$= \frac{4u_0}{\pi}\left[\frac{1}{\cosh(\pi b/a)} - \frac{1}{3\cosh(3\pi b/a)} + \frac{1}{5\cosh(5\pi b/a)} \cdots\right]$$

$\frac{11}{487}$ $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

Since it is circular $\therefore$ u is same for any θ $\therefore$ $u_\theta = 0$ and $u = f(r)$

$\therefore$ $u_{rr} + \frac{1}{r} u_r = 0$ $\therefore$ $u'' + \frac{u'}{r} = 0$, Let $u' = w$

$\therefore$ $w' + \frac{w}{r} = 0$

$\therefore$ $\frac{dw}{dr} = \frac{-w}{r}$ $\therefore$ $\int \frac{dw}{w} = \int -\frac{dr}{r}$ $\therefore$ $\ln w = -\ln r + c = \ln r^{-1} + c$

$\therefore$ $w = e^{\ln r^{-1} + c} = r^{-1} \cdot e^{c} = \frac{c'}{r}$

but $w = u'$

$\therefore$ $u' = \frac{c'}{r}$ $\therefore$ $\frac{du}{dr} = \frac{c'}{r}$ $\therefore$ $\int du = \int \frac{c'\, dr}{r}$

$\therefore$ $u = c' \ln r + c''$

But $u(a) = A$ and $u(b) = B$

$\therefore$ $\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} = \begin{bmatrix} c' \ln a + c'' \\ c' \ln b + c'' \end{bmatrix}$ $\Delta = \ln a - \ln b = \ln \frac{a}{b}$

$\Delta_{c'} = A - B$, $\Delta_{c''} = B \ln a - A \ln b$

$\therefore$ $c' = \frac{A - B}{\ln a - \ln b}$, $c'' = \frac{B \ln a - A \ln b}{\ln a - \ln b}$

$\therefore$ $u(r) = \frac{(A-B)\ln r + B \ln a - A \ln b}{\ln a - \ln b} = \frac{A(\ln r - \ln b) - B(\ln r - \ln a)}{\ln a - \ln b}$

$= \frac{A \ln\left(\frac{r}{b}\right) - B \ln\left(\frac{r}{a}\right)}{-(\ln b - \ln a)} = \frac{B \ln\left(\frac{r}{a}\right) - A \ln\left(\frac{r}{b}\right)}{\ln\left(\frac{b}{a}\right)}$

$\therefore$ $u(r) = \frac{B \ln(r/a) - A \ln(r/b)}{\ln(b/a)}$

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