بسم الله الرحمن الرحيم

الحلول المختارة لطلاب الهندسة والعمارة
المتغيرات المركبة

إعداد

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الأستاذ المشارك بقسم الهندسة الكهربية والحاسبات

بجامعة أم القرى

الطبعة الثانية

محرم ١٤٢٠هـ - نيسان ١٩٩٩

تمهيد

الحمد لله رب العالمين، والصلاة والسلام على سيد المرسلين، وعلى آله وصحبه أجمعين، وبعد: فهذه مجموعة من المسائل المحلولة في مادة المتغيرات المركبة (EE346) لطلبة قسم الهندسة الكهربية والحاسبات بكلية الهندسة والعمارة الإسلامية بجامعة أم القرى، إخترتها لجمعها المنهج المقرر، إبان تدريسي لهذه

المادة من كتاب: COMPLEX VARIABLES AND APPLICATIONS

للمؤلفين: R. V. CHURCHILL, J. W. BROWN & R. F. VERHEY

الطبعة الثالثة للطلاب في العام ١٩٧٤م، من دار النشر: McGRAW-HILL

وقد قمت بتبويبها وفهرستها لتسهل المراجعة فيها وتعم الفائدة منها واكتفيت عن ذكر المسألة بذكر رقمها والصفحة التي وردت فيها في الكتاب المذكور أعلاه والمقرر لمادة المتغيرات المركبة للمهندسين، وأخرجتها في أربع وثمانين صفحة في الطبعة الأولى عام ١٤٠٩هـ.

ثم زدت فيها في هذه الطبعة حصيلة أخرى على نفس النظام الأول، وساعدني مشكورا كل من المهندسين: خالد صدقة عتيق وماجد عثمان الزهراني في دمجها وتبويبها، كما قام المهندس: معيش مرزوق الحربي مشكورا بوضع جدول المقابلة بين الطبعة الثالثة المذكورة أعلاه والطبعة السادسة التي بين أيدي الطلاب.

والله أسأل أن ينفع بهذا العمل كل من يطالعه، وأن لا يحرمني ومن أعانني من مثوبته إنه سميع مجيب.

الفهرس

الباب	الفصل	أرقام المسائل	الصفحة
الثالث	٢١-٢٢	١أ، ١ب، ٣أ، ٣ب، ٣ج، ١٢،	٢٤
		١٦.	٢٥
	٢٣-٢٤	٦، ١٠أ،	٢٥
		١٠ب، ١٣،	٢٦
		١٤، ١٧أ،	٢٧
		١٧ب.	٢٨
	٢٥	٤،	٢٩
		٨، ٩ب،	٣٠
		٩ج.	٣١
	٢٦-٢٨	١أ، ٢ج، ٢د، ٣، ٩.	٣٢
	٢٩-٣٠	١أ، ١ب، ٢أ، ٢ب،	٣٣
		٢ج، ٦،	٣٤
		٨ب، ٨ج،	٣٥
		٨د، ١٠أ.	٣٦
الرابع	٣١-٣٢	٢، ٤،	٣٧
		٥،	٣٨
		٩، ١٤.	٤٠
	٣٣-٣٤	١،	٤١
		٢، ٣، ٧،	٤٣
		١١،	٤٤
		١٤.	٤٥
	٣٥-٣٧	١، ٢،	٤٦
		١٠.	٤٧
	٣٨-٤١	٣،	٤٨
		٧،	٥٠
		٨،	٥١
		١٠،	٥٢
		١٦،	٥٣
		١٨.	٥٤

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	5	3	5	2
	6	12	5	3
فقره d زیادەی 6	5	4	6	10
	5	2	6	16
	11	1	10	1
	11	2	10	3
	—	—	10	11
	11	10	10	12
	11	9	10	13
	12	18	11	15
	—	—	17	3
	17	5	17	9
	—	—	18	15
	25	1	21	1
	25	5	21	5
	25	7	21	7
	31	1	32	1
	—	—	32	2
	31	2	32	3

	رقم الصفحة	رقم المسألة	رقم الصفحة	رقم المسألة
	42	2	32	8
	47	1	39	3
	48	3	39	4
	48	(b) 8	39	8
	48	(a) 8	39	9
	62	1	49	1
	62	(c) 2	50	(b) 2
	62	4	50	3
	63	10	50	7
	63	12	50	11
	64	15	51	14
	67	1	55	1
	68	8	55	3
	67	3	56	12
	68	13	56	16
	72	8	59	6
	72	14	59	10
	72	17	59	13
	70	6	61	4

رقم السؤال	رقم الصفحة	رقم السؤال	رقم الصفحة	الملاحظات
8	61	9	74	
1	67	1	79	
2	67	2	79	
3	67	7	80	
9	67	4	79	إستخدام أ بدلأ من ع
1	71	1	84	
2	71	2	85	
7	71	15	85	
8	71	10	85	
10	71	11	85	
2	77	2	250	
4	77	4	250	
5	77	5	250	
9	77	10	250	
1	83	1	258	
2	83	2	258	
3	83	3	258	
7	83	7	259	
11	84	16	260	

الملاحظات	رقم الصفحة	رقم المسألة	رقم الصفحة	رقم المسألة
	260	15	84	14
	—	—	93	1
	—	—	93	2
	—	—	94	10
	266	4	100	3
	267	7	100	7
حل $\sin z$ في الثالث و e^z في السادس	266	3	100	8
	267	(b)9	101	10
	—	—	101	16
	268	13	101	18
	102	4	113	4
	102	7	113	6
	103	15	114	12
	103	(a,b)13	115	16
	120	4	126	1
	120	5	127	2
	119	1	127	5
	120	2	127	6
	128	1	138	3

[illegible]	رقم الصفحة	رقم المسألة	رقم الصفحة	رقم المسألة
	128	2	138	4
	136	6	139	10
	136	1	139	17
	142	1	144	1
	142	5	145	4
	–	–	149	2
	149	7	149	4
	150	11	149	6
	149	(b) 10	150	9
	–	–	161	3
	–	–	162	8
	157	5	168	4
	49	(c) 10	168	6
	–	–	168	9
	–	–	168	10
			179	1
[illegible]	197	1	179	2
[illegible]	198	11	180	3
	197	4	180	4
	197	5	180	5
	197	6	180	6
		.	180	7
	208	1	186	1
	208	5	186	5
	–	–	186	6
	214	1	186	9
	215	7	186	14
	215	9	186	15
	218	1	191 (a)	1 (a)
	219	2	191 (b)	1 (b)
	219	4	191	2
	219	5	191	4
	219	6	191	5
	219	7	191	6

الطبعة السابقة [illegible]

ملاحظات	رقم الصفحة	رقم المسألة	رقم الصفحة	رقم السطر
	–	–	192	8
	–	–	192	10
	–	–	194	15
	289	1	201	1
	289	2	201	2
	289	4	201	4
	–	–	201	5
	289	5	201	6
	297	1	207	
	297	5	207	6
	297	8	208	7
	312	2	224	2
	314	7	226	7

CENSTA 7mm

$\boxed{\frac{1d}{5}}$ ① $LHS = \frac{1+2i}{3-4i} + \frac{2-i}{5i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{(2-i)}{5i}\cdot\frac{i}{i} =$

$$= \frac{(3-8)+i(4+6)}{3^2+4^2} + \frac{2i+1}{-5} = \frac{-5+10i}{25} - \frac{1+2i}{5} =$$

$$= \frac{-1+\cancel{2i}-1-\cancel{2i}}{5} = \frac{-2}{5} = RHS \quad \therefore \text{ OK.}$$

$\boxed{\frac{1e}{5}}$ $\frac{5}{(1-i)(2-i)(3-i)} = \frac{5(1+i)(2+i)(3+i)}{(1^2-i^2)(2^2-i^2)(3^2-i^2)} = \frac{5(2-1+i(2+1))(3+i)}{(1+1)(4+1)(9+1)}$

$$= \frac{5(1+3i)(3+i)}{2*5*10} = \frac{3-3+i(9+1)}{20} = \frac{10i}{20} = \frac{i}{2}$$

$$\therefore \frac{5}{(1-i)(2-i)(3-i)} = \frac{i}{2} \qquad \therefore \text{ OK}$$

$\boxed{\frac{1f}{5}}$ ② $(1-i)^4 = [(1-i)^2]^2 = [1+i^2-2i]^2 = (1-1-2i)^2 = (-2i)^2 = 4i^2 = -4 < 0$

$\boxed{\frac{2}{5}}$ ① $z^2 - 2z + 2 = 0 \quad \therefore z = \frac{-(-2) \pm \sqrt{4-4(1)(2)}}{2} = \frac{2 \pm \sqrt{4-8}}{2}$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{-1}\cdot\sqrt{4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\therefore 1 \pm i$ are roots for $z^2 - 2z + 2 = 0$

<u>OK</u>

LHS (for $z = 1+i$) $= (1+i)^2 - 2(1+i) + 2 = \cancel{1} + \cancel{2i} + \cancel{i^2} - \cancel{2} - \cancel{2i} + \cancel{2} = 0$

LHS (for $z = 1-i$) $= (1-i)^2 - 2(1-i) + 2 = \cancel{1} - \cancel{2i} + \cancel{i^2} - \cancel{2} + \cancel{2i} + \cancel{2} = 0$

$\frac{2}{5}$ | $z + z + 1 = 0$

$\because \; z = x + iy$

$\therefore \; (x+iy)^2 + (x+iy) + 1 = 0 + i0$

or $\; x^2 - y^2 + 2xyi + x + iy + 1 = 0 + i0$

$\therefore \; x^2 - y^2 + x + 1 = 0 \quad (1) \quad \& \quad 2xy + y = 0 \quad (2)$

From (2) $\therefore \; y(2x+1) = 0 \quad \therefore y = 0$ or $x = -\frac{1}{2}$

If $y = 0 \quad \therefore$ (1) becomes $x^2 + x + 1 = 0 \quad \therefore x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$

This solution is rejected since x can not be complex but real

If $x = -\frac{1}{2} \quad \therefore$ (1) becomes $\frac{1}{4} - y^2 + (-\frac{1}{2}) + 1 = 0 \quad \therefore y^2 = \frac{3}{4} \quad \therefore y = \frac{\pm\sqrt{3}}{2}$

$\therefore$ The solution is $z = x + iy = -\frac{1}{2} + i\left(\frac{\pm\sqrt{3}}{2}\right) = \frac{-1 \pm \sqrt{3}\, i}{2}$

(Check: $z = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}$ $\therefore$ OK)

$\frac{10b}{6}$ | $Re(iz) = Re\left(i(x+iy)\right) = Re\left(ix - y\right) = -y$

$-Im\, z = -Im(x+iy) = -y$

$\therefore \; Re(iz) = -Im\, z \qquad \therefore$ OK.

$\frac{16}{6}$ | ① $LHS = (1+z)^2 = (1 + x + iy)^2 = (1+x)^2 + i^2y^2 + 2(1+x)yi =$

$= 1 + 2x + x^2 + i^2y^2 + 2yi + 2xyi =$

$= 1 + 2x + 2yi + x^2 + 2xyi + y^2 i^2 =$

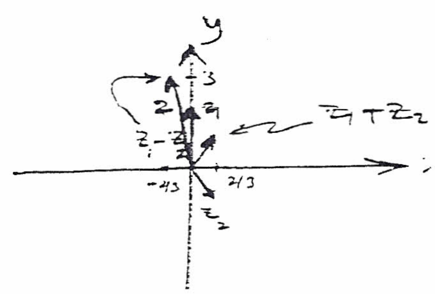
$= 1 + 2(x+iy) + (x+iy)^2 = 1 + 2z + z^2 = RHS$

$\therefore$ OK

$\frac{1a}{10}$ | $z_1 = 2i \;, \; z_2 = \frac{2}{3} - i$

$z_1 + z_2 = 2i + \frac{2}{3} - i = \frac{2}{3} + i$

$z_1 - z_2 = 2i - \frac{2}{3} + i = -\frac{2}{3} + 3i$



$\frac{3b}{10}$ $\quad \overline{iz} = -i\overline{z}$

$LHS = \overline{iz} = \overline{i(x+iy)} = \overline{ix-y} = \overline{-y+ix} = -y-ix$

$RHS = -i\overline{z} = -i\overline{(x+iy)} = -i(x-iy) = -ix-y = -y-i$

$\therefore$ $RHS = LHS$ $\therefore$ OK

$\frac{3c}{10}$ $\quad \frac{\overline{(2+i)^2}}{3-4i} = 1$

$LHS = \frac{\overline{(2+i)^2}}{3-4i} = \frac{\overline{4+4i+i^2}}{3-4i} = \frac{4-1-4i}{3-4i} = \frac{3-4i}{3-4i} = 1 = RHS$ $\therefore$

$\frac{3d}{10}$ ① $LHS = |(2\overline{z}+5)(\sqrt{2}-i)| = |2\overline{z}+5| \cdot |\sqrt{2}-i| = |2\overline{z}+5| \cdot \sqrt{2+1}$

$= \sqrt{3}\,|2\overline{z}+5| = \sqrt{3} \cdot \sqrt{(2\overline{z}+5)\overline{(2\overline{z}+5)}} =$

$= \sqrt{3} \cdot \sqrt{(2\overline{z}+5)(2z+5)}$

$RHS = \sqrt{3} \cdot \sqrt{(\cancel{2z}+5)\overline{(2z+5)}} = \sqrt{3} \cdot \sqrt{(2z+5)(2\overline{z}+5)} = LHS$ $\therefore$

\# Let $z = (-32)^{\frac{1}{5}}$

$\therefore$ $z^5 = -32 = 32\angle \pi+2k\pi$, k integer.

$\therefore$ $z = \left(32\angle (2k+1)\pi\right)^{\frac{1}{5}} = 2\angle \frac{2k+1}{5}\cdot\pi$

$\therefore$ Roots are:

$z_0 = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right) = 2(\cos 36^\circ + i\sin 36^\circ)$

$z_1 = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right) = 2(\cos 108^\circ + i\sin 108^\circ) = 2(-\cos 72^\circ + i\sin 72^\circ)$

$z_2 = 2\left(\cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}\right) = 2(-1+i0) = -2$

$z_3 = 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right) = 2(\cos 252^\circ + i\sin 252^\circ) = 2(-\cos 72^\circ - i\sin 72^\circ)$

$z_4 = 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right) = 2(\cos 324^\circ + i\sin 324^\circ) = 2(\cos 36^\circ - i\sin 36^\circ)$

i.e. $(-32)^{\frac{1}{5}} \in \{-2,\ 2(\cos 36^\circ \pm i\sin 36^\circ),\ 2(-\cos 72^\circ \pm i\sin 72^\circ)\}$

$\boxed{\frac{11b}{10}}$ $\big||z_1|-|z_2|\big| \leq |z_1-z_2| \leq |z_1|+|z_2|$

Let $z_1 = x_1+iy_1$ $\quad\therefore\ x_1^2+y_1^2 = |z_1|^2$

& Let $z_2 = x_2+iy_2$ $\quad\therefore\ x_2^2+y_2^2 = |z_2|^2$

$\therefore\ \big||z_1|-|z_2|\big|^2 = |z_1|^2-2|z_1|\cdot|z_2|+|z_2|^2 = x_1^2+y_1^2-2\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}+x_2^2+y_2^2$

$\therefore\ \big||z_1|-|z_2|\big|^2 - x_1^2-y_1^2-x_2^2-y_2^2 = -2\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}$ (1)

& $|z_1-z_2|^2 = |x_1-x_2+i(y_1-y_2)|^2 = (x_1-x_2)^2+(y_1-y_2)^2$

$= x_1^2+x_2^2-2x_1x_2+y_1^2+y_2^2-2y_1y_2$

$\therefore\ |z_1-z_2|^2 - x_1^2-y_1^2-x_2^2-y_2^2 = -2(x_1x_2+y_1y_2)$ (2)

& $(|z_1|+|z_2|)^2 = |z_1|^2+|z_2|^2+2|z_1||z_2| = x_1^2+y_1^2+x_2^2+y_2^2+2\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}$

$\therefore\ (|z_1|+|z_2|)^2 - x_1^2-y_1^2-x_2^2-y_2^2 = 2\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}$ (3)

Now, we have: $(x_1y_2-x_2y_1)^2 \geq 0 \Rightarrow x_1^2y_2^2+x_2^2y_1^2-2x_1y_2x_2y_1 \geq 0$

$\therefore\ x_1^2y_2^2+x_2^2y_1^2 \geq 2x_1x_2y_1y_2$

OR $(x_1^2x_2^2+y_1^2y_2^2)+x_1^2y_2^2+x_2^2y_1^2 \geq (x_1^2x_2^2+y_1^2y_2^2)+2x_1x_2y_1y_2$

$\therefore\ (x_1^2+y_1^2)\cdot(x_2^2+y_2^2) \geq (x_1x_2+y_1y_2)^2$

$\therefore\ \sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2} \geq \sqrt{(x_1x_2+y_1y_2)^2} = |x_1x_2+y_1y_2| \geq -(x_1x_2+y_1y_2)$

& $-\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2} \leq -(x_1x_2+y_1y_2)$ (5)

putting (4) & (5) together & multiplying by 2:

$\therefore\ 2\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2} \geq -2(x_1x_2+y_1y_2) \geq -2\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}$

$\therefore$ RHS of (3) $\geq$ RHS of (2) $\geq$ RHS of (1)

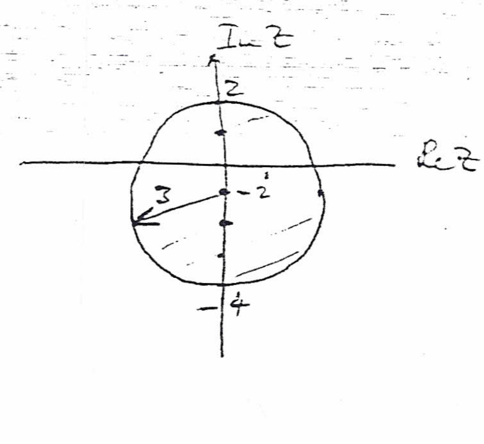
$\therefore$ LHS of (3) $\geq$ LHS of (2) $\geq$ LHS of (1)

$\therefore$ subtracting the common term among them gives:

$(|z_1|+|z_2|)^2 \geq |z_1-z_2|^2 \geq \big||z_1|-|z_2|\big|^2$. Taking the square root

$\therefore\ |z_1|+|z_2| \geq |z_1-z_2| \geq \big||z_1|-|z_2|\big|$ $\quad\therefore$ OK.

$\boxed{\frac{12b}{10}}$ $|z+i| \leq 3$



$|z+i| = 3$ is a circle centre $-i$, radius 3

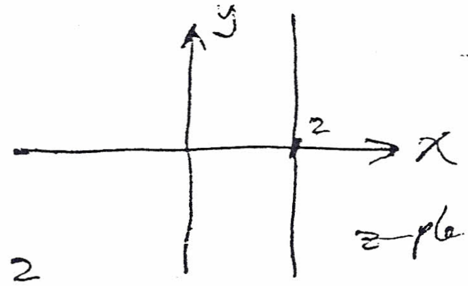
$\therefore\ |z+i| \leq 3$ is that circle & interior.

$\boxed{\frac{12c}{10}}$ $Re\left(\bar{z}-i\right)=2$

$\therefore$ $Re\left(x-iy-i\right)=2$

$\therefore$ $x=2$

$\therefore$ It is a line $\parallel$ to y crossing x at 2

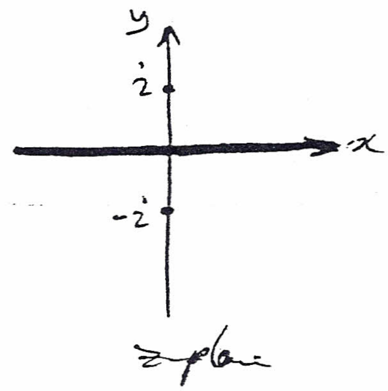


$\boxed{\frac{12d}{10}}$ ② $|z-i|=|z+i|$

$\therefore$ distance from i = distance from $-i$

$\therefore$ It is the line bisecting & $\perp$ to segment through $i, -i$.

$\therefore$ It is the x-axis.



OR:

$|x+iy-i|^2=|x+iy+i|^2 \Rightarrow \cancel{x^2}+(y-1)^2=\cancel{x^2}+(y+1)^2$

$\therefore$ $y^2-2y+1=y^2+2y+1$ $\quad\therefore -2y=2y \quad\therefore 4y=0 \quad\therefore y=0$ (x-axis

$\boxed{\frac{13}{10}}$ ① By triangle inequality $\therefore$ $|z_2+z_3|\geq\big||z_2|-|z_3|\big|$

Inverting $\therefore$ $\dfrac{1}{|z_2+z_3|}\leq\dfrac{1}{\big||z_2|-|z_3|\big|}$

Multiplying both sides by $|z_1|$ (a scalar):

$$\therefore \frac{|z_1|}{|z_2+z_3|}=\left|\frac{z_1}{z_2+z_3}\right|\leq\frac{|z_1|}{\big||z_2|-|z_3|\big|}, \quad \therefore \text{OK.}$$

$\boxed{\frac{15}{11}}$ ① $\therefore$ $x^2-y^2=1$ $\quad\therefore$ $\left(\dfrac{z+\bar{z}}{2}\right)^2-\left(\dfrac{z-\bar{z}}{2i}\right)^2=1$

$$\therefore \frac{z^2+\cancel{2z\bar{z}}+\bar{z}^2}{4}+\frac{z^2+\bar{z}^2-\cancel{2z\bar{z}}}{4}=1$$

$\therefore$ $z^2+\bar{z}^2=2$ $\quad$, $\therefore$ OK

$\boxed{\frac{2C}{17}}$ ③ $(-1+i)^7=\left(\sqrt{2}\angle\frac{3\pi}{4}\right)^7=(\sqrt{2})^7\cdot\angle\frac{3\pi}{4}*7=2^{\frac{7}{2}}\angle\frac{21\pi}{4}=2^{3+\frac{1}{2}}\angle 5\pi+\frac{\pi}{4}=$

$=8*\sqrt{2}\left(\cos(5\pi+\pi/4)+i\sin(5\pi+\pi/4)\right)=$

$=8\sqrt{2}\left(-\cos\pi/4-i\sin\pi/4\right)=8\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=-8(1+i)$ $\therefore$ ⓞ

$\frac{2d}{17}$

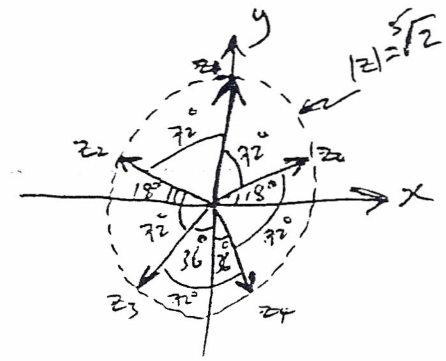
$$(1+i\sqrt{3})^{-10} = \left(2\angle\tfrac{\pi}{3}\right)^{-10} = 2^{-10}\angle\tfrac{-10\pi}{3} = 2^{-10}\left(\cos\left(\tfrac{-10\pi}{3}\right) + i\sin\left(\tfrac{-10\pi}{3}\right)\right) =$$

$$= 2^{-10}\left(\cos\left(\tfrac{10\pi}{3}\right) - i\sin\left(\tfrac{10\pi}{3}\right)\right) = 2^{-10}\left(\cos\left(\tfrac{4\pi}{3}\right) - i\sin\left(\tfrac{4\pi}{3}\right)\right) =$$

$$= 2^{-10}\left(-\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right) = 2^{-10}\left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) = 2^{-11}\left(-1+i\sqrt{3}\right) \therefore a$$

$\frac{3a}{17}$

Let $(2i)^{1/5} = z$



$\therefore\ z^5 = 2i = 2\angle\tfrac{\pi}{2}+2k\pi$

$\therefore\ z = 2^{\frac{1}{5}}\angle\tfrac{\pi}{10}+\tfrac{2k\pi}{5}$ $\quad k$ integer

$\therefore\ z_0 = 2^{\frac{1}{5}}\angle\pi/10 = 2^{\frac{1}{5}}\angle 18^\circ = \sqrt[5]{2}\left(\cos 18^\circ + i\sin 18^\circ\right)$

& $z_1 = 2^{\frac{1}{5}}\angle\tfrac{\pi}{10}+\tfrac{2\pi}{5} = 2^{\frac{1}{5}}\angle 18^\circ+72^\circ = \sqrt[5]{2}\left(\cos 90^\circ + i\sin 90^\circ\right) = i\sqrt[5]{2}$

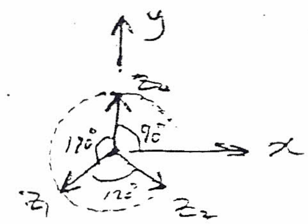
& $z_2 = 2^{\frac{1}{5}}\angle\tfrac{\pi}{10}+\tfrac{4\pi}{5} = 2^{\frac{1}{5}}\angle 18^\circ+144^\circ = \sqrt[5]{2}\left(\cos 162^\circ + i\sin 162^\circ\right) = \sqrt[5]{2}\left(-\cos 18^\circ + i\sin 18^\circ\right)$

& $z_3 = 2^{\frac{1}{5}}\angle\tfrac{\pi}{10}+\tfrac{6\pi}{5} = 2^{\frac{1}{5}}\angle 18^\circ+216^\circ = \sqrt[5]{2}\left(\cos 234^\circ + i\sin 234^\circ\right) = \sqrt[5]{2}\left(-\sin 36^\circ - i\cos 36^\circ\right)$

& $z_4 = 2^{\frac{1}{5}}\angle\tfrac{\pi}{10}+\tfrac{8\pi}{5} = 2^{\frac{1}{5}}\angle 18^\circ+288 = \sqrt[5]{2}\left(\cos 306^\circ + i\sin 306^\circ\right) = \sqrt[5]{2}\left(\sin 36^\circ - i\cos 36^\circ\right)$

$\frac{3b}{17}$

Let $(-i)^{1/3} = z$



$\therefore\ z^3 = -i = 1\angle\tfrac{3\pi}{2}+2k\pi$

$\therefore\ z = 1^{1/3}\angle\left(\tfrac{3\pi}{2}+2k\pi\right)/3$

$= 1\angle\tfrac{\pi}{2}+\tfrac{2k\pi}{3}$

$\therefore\ z_0$ (at $k=0$) $= 1\angle\tfrac{\pi}{2} = i$

& z_1 (at $k=1$) $= 1\angle\tfrac{\pi}{2}+\tfrac{2\pi}{3} = 1\angle 90^\circ+120^\circ = 1\angle 210^\circ = \cos 210^\circ + i\sin 210^\circ = \frac{-\sqrt{3}-i}{2}$

& z_2 (at $k=2$) $= 1\angle\tfrac{\pi}{2}+\tfrac{4\pi}{3} = 1\angle 90^\circ+240^\circ = 1\angle 330^\circ = \cos 330^\circ + i\sin 330^\circ = \frac{\sqrt{3}-i}{2}$

$\therefore$ The three roots are shown for $(-i)^{1/3} = i,\ \frac{\pm\sqrt{3}-i}{2}$.

3c/17 Let $z = (-1)^{1/3}$

$\therefore z^3 = -1 = 1\angle \pi + 2k\pi = 1\angle (1+2k)\pi$

$\therefore z = 1^{\frac{1}{3}} \angle \frac{(1+2k)\pi}{3} = 1\angle (1+2k)\pi/3 = \angle \pi/3, \angle \pi, \angle -\pi/3 =$

$= -1, \frac{1\pm\sqrt{3}i}{2}$

$\therefore$ Roots $(-1)^{1/3}$ are at $z=-1$, $z=(1\pm i\sqrt{3})/2$

3d/17 Let $8^{1/6}=z$ $\therefore z^6 = 8 = 8\angle 0+2k\pi \Rightarrow z=(8)^{1/6}\angle 2k\pi/6 = \sqrt{2}\angle$

$\therefore z_0 (\text{at } k=0) = \sqrt{2}\angle 0 = \sqrt{2}$

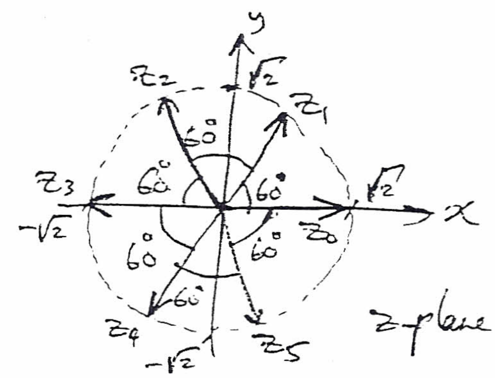
& $z_1 = \sqrt{2}\angle \frac{\pi}{3} = \sqrt{2}\left(\frac{1+\sqrt{3}i}{2}\right) = \frac{1+\sqrt{3}i}{\sqrt{2}}$

& $z_2 = \sqrt{2}\angle 2\pi/3 = \sqrt{2}\left(\frac{-1+\sqrt{3}i}{2}\right) = \frac{-1+\sqrt{3}i}{\sqrt{2}}$

& $z_3 = \sqrt{2}\angle \pi = -\sqrt{2}$

& $z_4 = \sqrt{2}\angle 4\pi/3 = \frac{-1-\sqrt{3}i}{\sqrt{2}}$

& $z_5 = \sqrt{2}\angle 5\pi/3 = \frac{1-\sqrt{3}i}{\sqrt{2}}$



$\therefore$ Roots of $8^{1/6}$ are $\pm\sqrt{2}$, $\frac{\pm1\pm\sqrt{3}i}{\sqrt{2}}$, shown above.

8/17 (b) $z^4+4=0$ $\therefore z^4 = -4 = 4\angle 180°+360°k$, k integer.

$\therefore z = (z^4)^{1/4} = (4\angle 180°+360°k)^{1/4} = \sqrt{2}\angle \frac{180+360k°}{4} = \sqrt{2}\angle 45+90k°$

$z_0 = \sqrt{2}\angle 45° = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1+i$

$z_1 = \sqrt{2}\angle 135° = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -1+i$

$z_2 = \sqrt{2}\angle 225° = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\left(\frac{-1}{\sqrt{2}}\right)\right) = -1-i$

$z_3 = \sqrt{2}\angle 315° = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 1-i$

$\therefore$ roots for $z^4+4=0$ are $\pm1\pm i$

$(z-z_0)(z-z_3) = z^2-(z_0+z_3)z+2 = z^2-2z+2$

& $(z-z_1)(z-z_2) = z^2+2z+2$

$\therefore z^4+4 = (z^2-2z+2)(z^2+2z+2)$

9/17 (a) $(e^{i\theta})^3 = e^{i3\theta}$ $\therefore (\cos\theta+i\sin\theta)^3 = \cos3\theta + i\sin3\theta$

$\therefore (\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta)(\cos\theta + i\sin\theta) = \cos3\theta + i\sin3\theta$

$\therefore \cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta + i(\cos^2\theta\sin\theta - \sin^3\theta + 2\sin\theta\cos^2\theta) = \cos3\theta + i\sin3\theta$

$\therefore \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos3\theta + i\sin3\theta$

$\therefore \cos3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ & $\sin3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

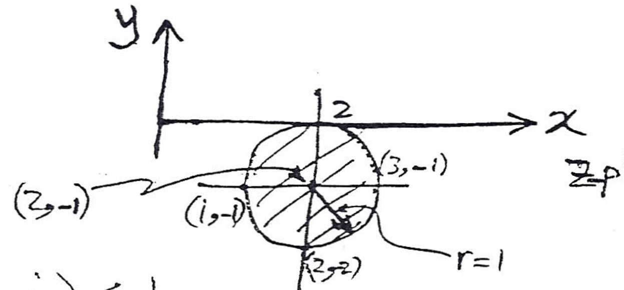
$\frac{15}{18}$ ① $\because$ z is nth root of unity $\therefore z = \sqrt[n]{1}$ $\therefore z^n = 1$

$\therefore z^n - 1 = 0$ $\therefore (z-1)(z^{n-1} + z^{n-2} + \cdots + 1) = 0$

$\because$ z is not unity $\therefore z - 1 \neq 0$

$\therefore z^{n-1} + z^{n-2} + \cdots + 1$ must be zero.

$\frac{1a}{21}$ $|z - 2 + i| \leq 1$



$\therefore |z - (2 - i)| \leq 1$

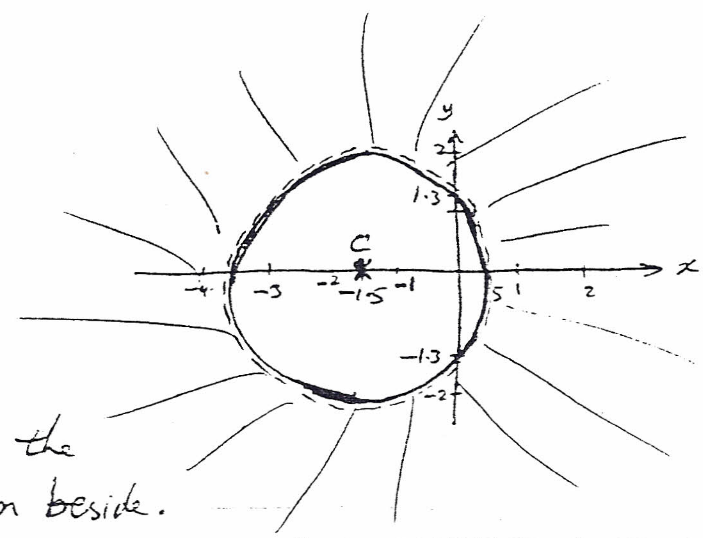
$\therefore$ distance between z & $(2-i) \leq 1$

$\therefore$ It is the circle with interior, whose centre is $(2,-1)$ radius

& since "domain" means open set that is connected

$\therefore$ The region is not a domain because it is connected but close.

$\frac{1b}{21}$ ② $|2z + 3| > 4$



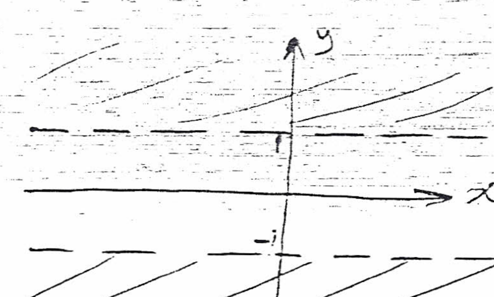
$\therefore |2(z + 1.5)| > 4$

$\therefore 2|z + 1.5| > 4$

$\therefore |z + 1.5| > 2$

$\therefore$ This region is the exterior of the circle $|z + 1.5| = 2$ as shown beside.

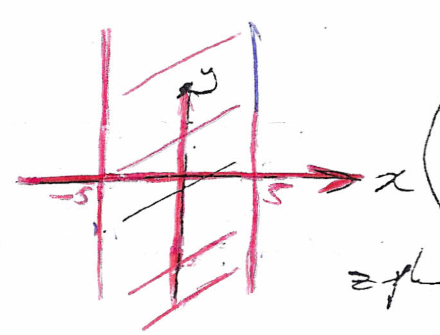
$\frac{1d}{21}$ $|\text{Im}\, z| > 1$



$\therefore |\text{Im}(x + iy)| > 1$

$\therefore |y| > 1$ $\therefore$ It is the region shown.

\# $|\text{Re}\, z| \leq 5 \Rightarrow |x| \leq 5$



$\therefore$ This is the region shown.

8

$\frac{1P}{21}$ ② $|z-4| \geq |z|$

$\therefore |z-4|^2 \geq |z|^2$

$\therefore |x+iy-4|^2 \geq |x+iy|^2$

$\therefore (x-4)^2+y^2 \geq x^2+y^2$

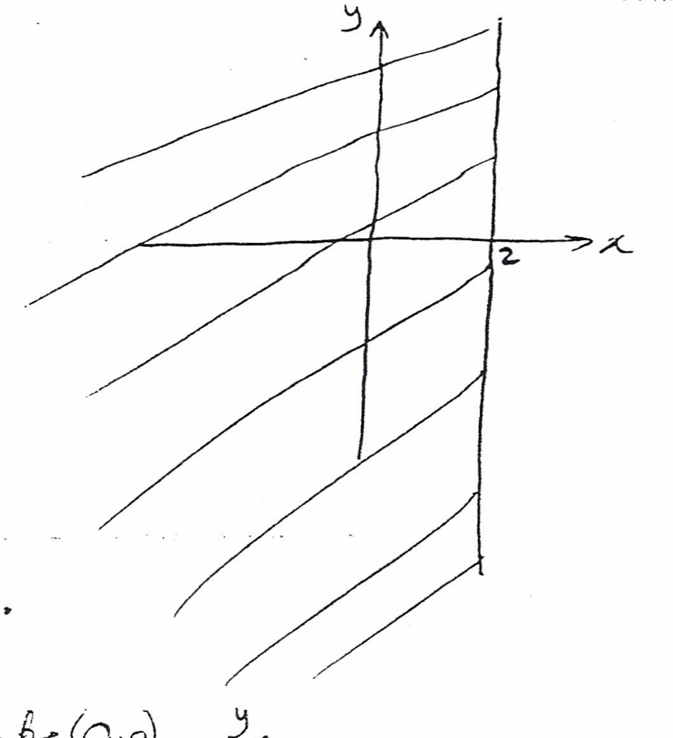
$\therefore x^2-8x+16 \geq x^2$

$\therefore 16 \geq 8x$

$\therefore 2 \geq x$

$\therefore x \leq 2$

$\therefore$ The above region is the line $x=2$ and the area to the left.

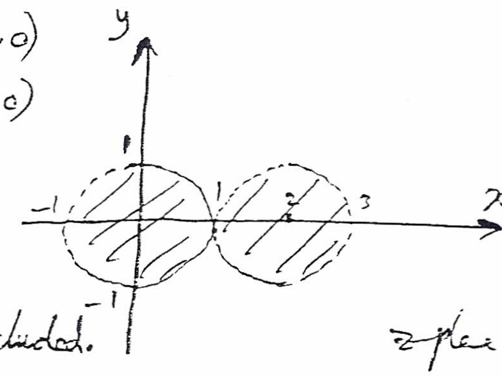


$\frac{5}{21}$ $|z|<1$ interior of unit circle centre $(0,0)$

$|z-2|<1$ " " " " at $(2,0)$

$\therefore$ S is the shadowed area.

$\therefore$ S is not connected because both circles are not intersecting since the boundaries of both are excluded.



$\frac{7a}{21}$ ① $z_n = i^n$

$\therefore$ z_n can be seen to have one of the following values $1, i, -1, -i$, hence if you chose z_0 to be one of them you will find that no point of them lie in its ϵ-N $\therefore$ None of them is an accumulation point

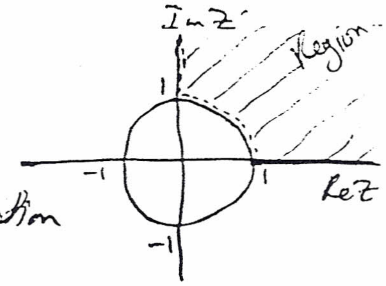
$\therefore$ $z_n = i^n$ has no accumulation point.

$\frac{7b}{21}$ ① $z_n = \left(\frac{1}{n}\right) \cdot i^n$ $\therefore$ The set of z_n is $\{i, -\frac{1}{2}, -\frac{i}{3}, \frac{1}{4}, \frac{i}{5}, \ldots$

Choosing $z_0 = 0$, we can be sure to have at least one point of the set in its ϵ-N, e.g. if $\epsilon = .00001$, then the points for $n > 10^5$ are in the ϵ-N.

$\therefore$ 0 is the accumulation point.

$\boxed{\frac{7c}{21}}$ ③ $|z|>1, 0 \le \arg z < \pi/2$



From graph, it is seen that the accumulation points of the set are:

$y=0$ & $1 \le x < \infty$ (x-axis from 1 on)

, $x^2+y^2=1$ & $x \geq 0$ & $y \geq 0$ (positive quarter of unit circle,

& $x=0$ & $1 \le y \le \infty$ (y-axis from 1 on)

$\boxed{\frac{7d}{21}}$ ② $z_n = (-1)^n (1+i)\frac{(n-1)}{n}$ $\therefore$ S is $\{0, \frac{1+i}{2}, -\frac{2}{3}(1+i), \frac{3}{4}(1+i), -\frac{4}{5}(1+i), \ldots$

It can be seen that S may split into positive and negative terms of $(1+i)$ with coeffecient of $\frac{n-1}{n}$ that approches unity as $n \to \infty$

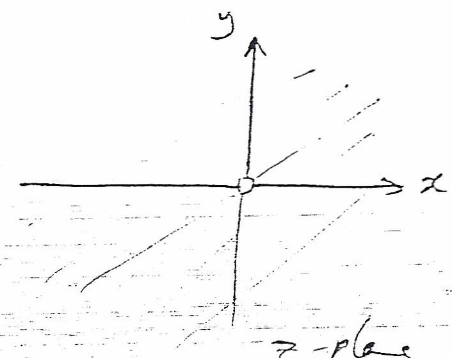
$\therefore$ We have two accumulation points $\pm(1+i)$ because $(-1)^{10000}(1+i)\left(\frac{9999}{10000}\right)$ and $(-1)^{10001}(1+i)\left(\frac{10000}{10001}\right)$ lie in the .0003-N of these points.

$\boxed{\frac{1a}{32}}$ $f(z) = \dfrac{1}{z^2+1}$ domain is all z: $z^2+1 \neq 0$ i.e $z^2 \neq -1$

i.e $z^2 \neq i^2$, i.e $z \neq \pm i$

$\boxed{\frac{1b}{32}}$ $f(z) = \text{Arg}\left(\frac{1}{z}\right) = \text{Arg}(1) - \text{Arg}(z)$

$= 2n\pi - \text{Arg}(z)$



When $z=0$, $\text{Arg}(z)$ is undefined

$\therefore$ domain is $z \neq 0$ as shown.

$\boxed{\frac{c}{32}}$ ① Domain of $f(z) = \dfrac{z}{z+\bar{z}}$ is $z+\bar{z} \neq 0$

i.e $x+iy+x-iy \neq 0$ i.e $2x \neq 0$ i.e $x \neq 0$

$\therefore$ The domain is the entire complex plane without the y-axis

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$\frac{1d}{32}$ ① $f(z) = \dfrac{1}{1-|z|^2}$ $\therefore$ Domain is $1-|z|^2 \neq 0$

$\therefore |z|^2 \neq 1$

$\therefore |z| \neq 1$

$\therefore$ The domain is the entire complex plane but not the points of the circumference of the unit circle centred at the origin

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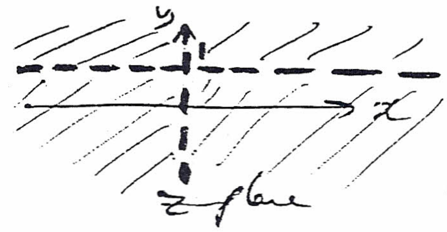
$\frac{2}{32}$ ① Domain of definition of $g(z) = \dfrac{y}{x} + \dfrac{i}{1-y}$ is all the z-plane

except the singularity points at $x=0$ or $1-y=0 \Rightarrow y=1$

$\therefore$ The domain is as shown.



$\frac{3}{32}$ ② $f(z) = z^3 + z + 1 = (x+iy)^3 + (x+iy) + 1 =$

$= x^3 + 3x^2 iy + 3x i^2 y^2 + i^3 y^3 + x + iy + 1 =$

$= (x^3 - 3xy^2 + x + 1) + i(3x^2 y - y^3 + y) = u + iv$

$\therefore u(x,y) = x^3 + x - 3xy^2 + 1$

$\&$ $v(x,y) = 3x^2 y - y^3 + y$

$\frac{8b}{32}$ ① $\lim\limits_{z\to i} \dfrac{iz^3 - 1}{z+i} = \dfrac{i \cdot i^3 - 1}{i+i} = \dfrac{1-1}{2i} = 0$

$\frac{12}{33}$ ③ ⓐ $\lim\limits_{z\to\infty} \dfrac{1}{z^2+1} = \lim\limits_{\frac{1}{z}\to 0} \dfrac{(1/z^2)}{\left(\frac{z^2+1}{z^2}\right)} = \lim\limits_{\frac{1}{z}\to 0} \dfrac{\left(\frac{1}{z}\right)^2}{1+\left(\frac{1}{z}\right)^2} = \dfrac{0}{1+0} = 0$

ⓑ $\lim\limits_{z\to 1} \dfrac{1}{(z-1)^3} = \dfrac{1}{\lim\limits_{z\to 1}(z-1)^3} = \dfrac{1}{(1-1)^3} = \dfrac{1}{0} = \infty$

ⓒ $\lim\limits_{z\to\infty} 3z^2 = \lim\limits_{\frac{1}{z}\to 0} 3z^2 = \lim\limits_{\frac{1}{z}\to 0} \dfrac{3}{\left(\frac{1}{z}\right)^2} = \dfrac{3}{0} = \infty$

3/39 (4)

(a) $f(z) = 3z^2 - 2z + 4 \quad \therefore f'(z) = 6z - 2$

(b) $f(z) = (1-4z^2)^3 \quad \therefore f'(z) = 3(1-4z^2)^2(-8z) = -24z(1-4z^2)^2$

(c) $f(z) = \frac{z-1}{2z+1} \quad \therefore f'(z) = \frac{2z+1-2(z-1)}{(2z+1)^2} = \frac{3}{(2z+1)^2}, \quad z \neq -\frac{1}{2}$

(d) $f(z) = \frac{(1+z^2)^4}{z^2} \quad \therefore f'(z) = \frac{4(1+z^2)^3 * 2z * z^2 - 2z * (1+z^2)^4}{z^4} =$

$= \frac{(1+z^2)^3}{z^4} \cdot \left(8z^3 - 2z(1+z^2)\right) = \frac{(1+z^2)^3}{z^4} \cdot (6z^3 - 2z) =$

$= \frac{2(1+z^2)^3 \cdot (3z^2-1)}{z^3}, \quad z \neq 0$

4/39

$f(z) = \frac{1}{z}$ find $f'(z)$ by $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$\therefore f'(z) = \lim_{\Delta z \to 0} \left[\left(\frac{1}{z+\Delta z} - \frac{1}{z}\right)/\Delta z\right] = \lim_{\Delta z \to 0} \frac{z-(z+\Delta z)}{z \Delta z (z+\Delta z)} =$

$= \lim_{\Delta z \to 0} \frac{-1}{z(z+\Delta z)} = \frac{-1}{z^2} \qquad z \neq 0 \qquad \therefore \text{OK}$

$\boxed{\frac{8}{39}}$ $f(z) = \operatorname{Re} z$

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\operatorname{Re}(z+\Delta z) - \operatorname{Re} z}{\Delta x + i\Delta y} =$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\operatorname{Re}(x+iy+\Delta x+i\Delta y) - \operatorname{Re}(x+iy)}{\Delta x + i\Delta y} =$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x+\Delta x - x}{\Delta x + i\Delta y} = \frac{\Delta x}{\Delta x + i\Delta y}$$

Consider the case when we approach z horizontally

$\therefore \Delta y = 0$

$$\therefore f'(z) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x + i0} = 1$$

Now, approaching z vertically gives $\Delta x = 0$

$$\therefore f'(z) = \lim_{\Delta y \to 0} \frac{0}{0 + i\Delta y} = 0$$

& Approaching along $y = x$ $\therefore \Delta y = \Delta x$

$$\therefore f'(z) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x + i\Delta x} = \frac{1}{1+i}$$

$\therefore f(z)$ is not differentiable anywhere since the limit does not exist.

$\boxed{\frac{9}{39}}$ ② $f(z) = \bar{z}$ $\therefore f'(z) = \lim_{\Delta z \to 0} \dfrac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \dfrac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} =$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \begin{cases} \Delta z \text{ real: } \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1 \\ \Delta z \text{ imaginary: } \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1 \end{cases}$$

$\therefore f'(z)$ has no limit $\therefore$ $f(z)$ is not differentiable anywhere

$\boxed{\frac{1a}{44}}$ ② $f(z) = \bar{z}$ $\therefore f(x,y) = x - iy = u + iv$ $\therefore u(x,y) = x$ & $v(x,y) = -y$

$\therefore u_x = 1$, $v_x = 0$, $u_y = 0$, $v_y = -1$

$\therefore u_x \neq v_y$ (because $1 \neq -1$) $\therefore$ Cauchy is not happy

$\therefore f(z) = \bar{z}$ is not differentiable anywhere.

$\boxed{\frac{1b}{44}}$ $f(z) = z - \bar{z} = x + iy - (x - iy) = 2iy = u + iv$

$\therefore u = 0$ & $v = 2y$

$\therefore u_x = u_y = 0$ & $v_x = 0$, $v_y = 2$

$u_y = -v_x$ $\therefore$ OK
but $u_x = 0$ & $v_y = 2$ & $0 \neq 2$ $\therefore$ Cauchy is unhappy at all

$\therefore f(z)$ is nowhere differentiable.

$\boxed{\frac{1c}{44}}$ $f(z) = 2x + ixy^2$ $\therefore u = 2x$ & $v = xy^2$

$\therefore u_x = 2$, $u_y = 0$ & $v_x = y^2$ & $v_y = 2xy$

$\therefore u_x = v_y \iff 2 = 2xy \iff xy = 1$ (1)
& $u_y = -v_x \iff 0 = -y^2 \iff y = 0$ (2)
Conditions (1) & (2) are contradicting $\therefore f$ is nowhere differentiable

$\boxed{\frac{1d}{44}}$ ② $f(z) = e^x \cdot e^{-iy} = e^x(\cos(-y) + i\sin(-y)) = e^x(\cos y - i\sin y) =$
$= e^x\cos y + i(-e^x\sin y) = u(x,y) + i\,v(x,y) \Rightarrow$
$\therefore u(x,y) = e^x\cos y$ & $v(x,y) = -e^x\sin y$
$\therefore u_x = e^x\cos y$, $u_y = -e^x\sin y$, $v_x = -e^x\sin y$ & $v_y = -e^x\cos y$
$\therefore u_x \neq v_y$ unless $e^x\cos y = -e^x\cos y$ i.e. $y = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, (2n+1)\frac{\pi}{2}$ n integer
but then $u_y \neq -v_x$ $\therefore$ Cauchy is not happy $\therefore f(z)$ is nowhere differentiable.

$\frac{2a}{45}$ (6) $f(z) = iz+2 = i(x+iy)+2 = 2-y+ix = u_1(x,y)+i\,v_1(x,y)$

$\therefore u_1(x,y) = 2-y$ & $v_1(x,y) = x$

$\therefore u_{1x} = 0$, $u_{1y} = -1$, $v_{1x} = 1$, $v_{1y} = 0$

$\therefore u_{1x} = 0 = v_{1y}$ & $u_{1y} = -1 = -(1) = -v_{1x}$ $\therefore$ Cauchy OK

$\therefore f(z)$ is differentiable everywhere.

$\therefore f'(z) = u_x + i v_x = 0 + i = i$ $\therefore f'(z) = i$

Let $g(z) = f'(z) = i = u_2(x,y) + i v_2(x,y)$

$\therefore u_2(x,y) = 0$ & $v_2(x,y) = 1$

$\therefore u_{2x} = 0$, $u_{2y} = 0$, $v_{2x} = 0$, $v_{2y} = 0$

$\therefore u_{2x} = 0 = v_{2y}$ & $u_{2y} = 0 = -(0) = -v_{2x}$ $\therefore$ Cauchy OK

$\therefore g(z) = f'(z)$ is differentiable everywhere.

$\therefore g'(z) = f''(z) = u_{2x} + i v_{2x} = 0 + i0 = 0$ $\therefore f''(z) = 0$

$\frac{2d}{45}$ (6) $f(z) = \cos x \cosh y - i \sin x \sinh y = u_1(x,y) + i v_1(x,y)$

$\therefore u_1(x,y) = \cos x \cosh y$ & $v_1(x,y) = -\sin x \sinh y$

$\therefore u_{1x} = -\sin x \cosh y$, $u_{1y} = \cos x \sinh y$, $v_{1x} = -\cos x \sinh y$, $v_{1y} = -\sin x \cosh y$

$\therefore u_{1x} = v_{1y}$ & $u_{1y} = -v_{1x}$ $\therefore$ Cauchy OK everywhere

$\therefore f(z)$ is differentiable everywhere.

$\therefore f'(z) = u_{1x} + i v_{1x} = -\sin x \cosh y - i \cos x \sinh y$

Let $g(z) = f'(z) = -\sin x \cosh y - i \cos x \sinh y = u_2(x,y) + i v_2(x,y)$

$\therefore u_2(x,y) = -\sin x \cosh y$ & $v_2 = -\cos x \sinh y$

$\therefore u_{2x} = -\cos x \cosh y$, $u_{2y} = -\sin x \sinh y$, $v_{2x} = \sin x \sinh y$, $v_{2y} = -\cos x \cosh y$

$\therefore u_{2x} = v_{2y}$ & $u_{2y} = -v_{2x}$ $\therefore$ Cauchy is OK everywhere

$\therefore g(z) = f'(z)$ is differentiable everywhere.

$\therefore g'(z) = f''(z) = u_{2x} + i v_{2x} = -\cos x \cosh y + i \sin x \sinh y = -f(z)$

$\boxed{\dfrac{3C}{45}}$ $f(z) = z\,\mathrm{Im}\,z = (x+iy)\cdot y = xy + iy^2 = u + iv$

$\therefore$ $u(x,y) = xy$ & $v(x,y) = y^2$

$\therefore$ $u_x = y$, $u_y = x$, $v_x = 0$ & $v_y = 2y$

$\because$ $u_x = v_y \Leftrightarrow y = 2y \Leftrightarrow y = 0$

& $u_y = -v_x \Leftrightarrow x = -0 \Leftrightarrow x = 0$

$\therefore$ Cauchy is happy only at the point $(0,0) = 0 + i0 = 0$

$\therefore$ $f'(0) = u_x + iv_x\Big|_{z=0} = y + i0\Big|_{y=0} = 0$

$\boxed{\#}$ $u(x,y) = e^x \cos y$

$\therefore$ $u_x = e^x \cos y$, $u_{xx} = e^x \cos y$

& $u_y = -e^x \sin y$ & $u_{yy} = -e^x \cos y$

$\therefore$ $u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$ $\therefore$ u is harmonic.

Let its conjugate be v:

$\therefore$ $v_x = -u_y = e^x \sin y \Rightarrow v = \int e^x \sin y \,\partial x = e^x \sin y + f(y)$

$\therefore$ $v_y = e^x \cos y + f'(y)$

but, $v_y = u_x = e^x \cos y \Rightarrow f'(y) = 0$ $\therefore f(y) = C$

$\therefore$ $v(x,y) = e^x \sin y + C$ is the harmonic conjugate.

$\frac{8}{45}$ ② $u_x = u_r \cos\theta - \frac{1}{r} u_\theta \sin\theta$ & $u_y = u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta$

$\therefore$ Similarly:

$V_x = V_r \cos\theta - \frac{1}{r} V_\theta \sin\theta$ & $V_y = V_r \sin\theta + \frac{1}{r} V_\theta \cos\theta$

According to Rectangular Cauchy $\therefore$ $\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} V_y \\ -V_x \end{bmatrix}$.

Using the above substitutions for u_x, u_y, V_x, V_y in matrix form:

$$\therefore \begin{bmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & \frac{\cos\theta}{r} \\ -\cos\theta & \frac{\sin\theta}{r} \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sin\theta & \frac{\cos\theta}{r} \\ -\cos\theta & \frac{\sin\theta}{r} \end{bmatrix} \cdot \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} =$$

$$= \frac{1}{\frac{\cos^2\theta}{r} + \frac{\sin^2\theta}{r}} \cdot \begin{bmatrix} \frac{\cos\theta}{r} & \frac{\sin\theta}{r} \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \sin\theta & \frac{\cos\theta}{r} \\ -\cos\theta & \frac{\sin\theta}{r} \end{bmatrix} \cdot \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} =$$

$$= r \cdot \begin{bmatrix} \frac{\cos\theta\sin\theta}{r} - \frac{\sin\theta\cos\theta}{r} & \frac{\cos^2\theta}{r^2} + \frac{\sin^2\theta}{r^2} \\ -\sin^2\theta - \cos^2\theta & -\frac{\sin\theta\cos\theta}{r} + \frac{\cos\theta\sin\theta}{r} \end{bmatrix} \cdot \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} =$$

$$= r \begin{bmatrix} 0 & \frac{1}{r^2} \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{r} \\ -r & 0 \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \begin{bmatrix} \frac{V_\theta}{r} \\ -rV_r \end{bmatrix} = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix}$$

$\therefore$ $u_r = \frac{V_\theta}{r}$ & $u_\theta = -rV_r$

Hence, Cauchy Polar Equations are:

$u_r = \frac{V_\theta}{r}$ & $\underline{\frac{u_\theta}{r} = -V_r}$

1b/49 $u(x,y) = \sin x \cosh y$

$\therefore$ $u_x = \cos x \cosh y$ & $u_{xx} = -\sin x \cosh y$

& $u_y = \sin x \sinh y$ & $u_{yy} = \sin x \cosh y$

$\therefore$ $u_{xx} + u_{yy} = -\sin x \cosh y + \sin x \cosh y = 0$ $\therefore$ u is harmonic.

Let its conjugate be v

$\because$ $v_y = u_x = \cos x \cosh y$ $\therefore$ $v = \int \cos x \cosh y \, dy = \cos x \sinh y + g(x)$

$\therefore$ $v_x = -\sin x \sinh y + g'(x) = -u_y = -\sin x \sinh y$ $\therefore$ $g'(x) = 0$

$\therefore$ $g(x) =$ constant c

$\therefore$ $v = \cos x \sinh y + c$ is the harmonic conjugate for u.

$f(z) = \sin x \cosh y + i \cos x \sinh y = u + iv$

To prove that a function is entire, we need to prove that the Cauchy-Riemann equations (CRE) hold throughout z-plane.

$\therefore$ $u = \sin x \cosh y$, $v = \cos x \sinh y$

$\therefore$ $u_x = \cos x \cosh y$, $v_y = \cos x \cosh y$ $\therefore$ $u_x = v_y$, always

& $u_y = \sin x \sinh y$, $v_x = -\sin x \sinh y$ $\therefore$ $u_y = -v_x$, always

$\therefore$ $f(z)$ is entire

1c/49 $f(z) = e^{-y} e^{ix} = e^{-y}(\cos x + i \sin x) = e^{-y}\cos x + i e^{-y} \sin x = u + iv$

$\therefore$ $u(x,y) = e^{-y}\cos x$ & $u_x = -e^{-y}\sin x$ & $u_y = -e^{-y}\cos x$

& $v(x,y) = e^{-y}\sin x$ & $v_x = e^{-y}\cos x$ & $v_y = -e^{-y}\sin x$

$\therefore$ $u_x = -e^{-y}\sin x = v_y$ & $u_y = -e^{-y}\cos x = -v_x$ $\therefore$ Cauchy is always happy $\therefore$ differentiable everywhere $\therefore$ entire.

1d/49 ② $f(z) = (z^2 - 2) \cdot e^{-x} e^{-iy} = (z^2 - 2) \cdot e^{-x-iy} = (z^2-2) e^{-(x+iy)} = (z^2-2) e^{-z}$

$\because$ z-polynomials are entire $\therefore$ $(z^2 - 2)$ is entire

$\because$ z-exponentials are entire $\therefore$ e^{-z} is entire

$\because$ Composition of entire functions is entire $\therefore$ $(z^2-2) \cdot e^{-z}$ is entire

$\therefore$ $f(z)$ is entire.

$\frac{2}{50}$ (a) (2) $f(z) = xy + iy = u + iv$

$\therefore$ $u = xy$, $v = y$

$\therefore$ $u_x = y$, $v_y = 1$ $\therefore u_x = v_y$ only when $y=1$

& $u_y = x$, $v_x = 0$ $\therefore u_y = -v_x$ only when $x=0$

$\therefore$ The function satisfy CRE only at the point $(0,1)$

$\because$ Analyticity at z_0 implies differentiability at z_0 and its neighborhood

$\because$ $f(z)$ is only differentiable at $(0,1)$ no where else

$\therefore$ $f(z)$ is not analytic anywhere.

(2)
(b) $f(z) = e^y \cdot e^{ix} = e^y(\cos x + i\sin x) = e^y\cos x + i e^y \sin x = u + iv$

$\therefore$ $u = e^y \cos x$, $v = e^y \sin x$

$\therefore$ $u_x = -e^y \sin x$, $v_y = e^y \sin x$ $\therefore u_x = v_y$ only when $x = n\pi$, $n = 0, \pm 1, \ldots$

& $u_y = e^y \cos x$, $v_x = e^y \cos x$ $\therefore u_y = -v_x$ only when $x = (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$

$\therefore$ To satisfy one equation of CRE the other is not satisfyed.

$\therefore$ The function doesn't satisfy CRE anywhere

$\therefore$ $f(z)$ is not analytic anywhere.

$\frac{3a}{50}$ (3) $\dfrac{2z+1}{z(z^2+1)}$ singularity at $z(z^2+1) = 0$ or $z = 0, \pm i$

$\frac{7a}{50}$ (5) $u = 2x(1-y)$, $v = ?$

$\therefore$ $u_x = 2(1-y)$, $u_{xx} = 0$

& $u_y = -2x$, $u_{yy} = 0$

$\therefore$ $u_{xx} + u_{yy} = 0 + 0 = 0$ $\therefore$ u is harmonic throughout z

$\because$ $u_x = v_y$

$\therefore$ $v_y = 2(1-y)$ $\therefore$ $v = \int 2(1-y)\,\partial y = 2y - y^2 + g(x)$

$\because$ $u_y = -v_x$

$\therefore$ $-2x = -v_x = -(0 + g'(x))$ $\Rightarrow$ $g'(x) = 2x$

$\therefore$ $g(x) = \int 2x\,dx = x^2 + C$, C = constant

$\therefore$ $v = 2y - y^2 + g(x) = 2y - y^2 + x^2 + C$

$\therefore$ The harmonic conjugate of u is $v = x^2 - y^2 + 2y + C$

$\boxed{\frac{7b}{50}}$ (4) $u = 2x - x^3 + 3xy^2$

$\therefore\ u_x = 2 - 3x^2 + 3y^2$ & $u_{xx} = -6x$

& $u_y = 6xy$ & $u_{yy} = 6x$

$\therefore\ u_{xx} + u_{yy} = 0$ $\therefore\ u$ is harmonic

$\because\ v$ is a harmonic conjugate of u $\therefore\ u_x = v_y = 2 - 3x^2 + 3y^2$

$\therefore\ v = \int (2 - 3x^2 + 3y^2)\,\partial y = 2y - 3x^2y + y^3 + g(x)$

$\therefore\ v_x = -6xy + g'(x) = -u_y = -6xy$ $\therefore\ g'(x) = 0$ $\therefore\ g(x) = c$

$\therefore$ The harmonic conjugate, $v = 2y - 3x^2y + y^3 + C$

$\boxed{\frac{7c}{50}}$ $u = \sinh x \sin y$

(1) $\therefore\ u_x = \cosh x \sin y$, $u_{xx} = \sinh x \sin y$

(1) & $u_y = \sinh x \cos y$, $u_{yy} = -\sinh x \sin y$

(1) $\therefore\ u_{xx} + u_{yy} = 0$ $\therefore\ u$ is harmonic.

(1) $\because\ u_x = v_y$ $\therefore\ v_y = \cosh x \sin y$ $\therefore\ v = \int \cosh x \sin y\,\partial y = -\cosh x \cos y + g(x)$

$\therefore\ v_x = -\sinh x \cos y + g'(x)$

but $v_x = -u_y = -\sinh x \cos y$

(1) $\therefore\ -\sinh x \cos y + g'(x) = -\sinh x \cos y$ $\therefore\ g'(x) = 0$ $\therefore\ g(x) = c$

(2) $\therefore\ v = -\cosh x \cos y + C$ is the harmonic conjugate of u

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$u(x,y) = y^3 - 3x^2y$

$\therefore\ u_x = -6xy$ & $u_{xx} = -6y$

& $u_y = 3y^2 - 3x^2$ & $u_{yy} = 6y$

$\therefore\ u_{xx} + u_{yy} = -6y + 6y = 0$ $\therefore\ u$ is harmonic $\therefore$ OK

Let the harmonic conjugate be v.

$\therefore\ v_y = u_x = -6xy$ $\therefore\ v = \int -6xy\,\partial y = -3xy^2 + f(x)$

and since $v_x = -u_y = -(3y^2 - 3x^2) = 3x^2 - 3y^2 = -3y^2 + f'(x)$

$\therefore\ f'(x) = 3x^2$ $\therefore\ f(x) = x^3 + c$

$\therefore\ v(x,y) = -3xy^2 + f(x) = x^3 - 3xy^2 + c$ is the harmonic conjugate.

$\frac{7d}{50}$ (5) $u = \frac{y}{x^2+y^2}$, $v = ?$

$\therefore u_x = \frac{-2xy}{(x^2+y^2)^2}$, $u_{xx} = \frac{-2y(x^2+y^2)^2 + 2xy * 2(x^2+y^2)(2x)}{(x^2+y^2)^4} =$

$= \frac{-2yx^2 - 2y^3 + 8x^2y}{(x^2+y^2)^3} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$

& $u_y = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$, $u_{yy} = \frac{-2y(x^2+y^2)^2 - 2(x^2+y^2)(2y)(x^2-y^2)}{(x^2+y^2)^4} =$

$= \frac{-2yx^2-2y^3-4yx^2+4y^3}{(x^2+y^2)^3} = \frac{-6yx^2+2y^3}{(x^2+y^2)^3} = \frac{-2y(3x^2-y^2)}{(x^2+y^2)^3} = -u_{xx}$

$\therefore u_{xx} + u_{yy} = 0$ $\therefore$ u is harmonic throughout the z-plane.

$\because u_x = v_y$

$\therefore v_y = \frac{-2xy}{(x^2+y^2)^2}$ $\therefore v = \int \frac{-2xy}{(x^2+y^2)^2}\,\partial y = \frac{x}{x^2+y^2} + g(x)$

$\because u_y = -v_x$

$\therefore \frac{x^2-y^2}{(x^2+y^2)^2} = -v_x = -\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + g'(x)\right] = \frac{x^2-y^2}{(x^2+y^2)^2} - g'(x) \Rightarrow$

$\therefore g'(x) = 0$ $\therefore g(x) = \int 0\,dx = 0 + C$ $\therefore g(x) = C$

$\therefore v = \frac{x}{x^2+y^2} + C$

$\therefore$ The harmonic conjugate of u is $v = \frac{x}{x^2+y^2} + C$.

$u(r,\theta) = \frac{\sin\theta}{r}$ $\therefore u_r = \frac{-\sin\theta}{r^2}$, $u_{rr} = \frac{2\sin\theta}{r^3}$

& $u_\theta = \frac{\cos\theta}{r}$, $u_{\theta\theta} = \frac{-\sin\theta}{r}$

$\therefore r^2 u_{rr} + r u_r + u_{\theta\theta} = \frac{2\sin\theta}{r} - \frac{\sin\theta}{r} - \frac{\sin\theta}{r} = 0$ $\therefore u$ is harmonic.

To find harmonic conjugate v, $\because v_\theta = r u_r = \frac{-\sin\theta}{r}$

$\therefore v = \int \frac{-\sin\theta}{r}\,\partial\theta = \frac{\cos\theta}{r} + f(r)$ $\therefore v_r = \frac{-\cos\theta}{r^2} + f'(r) = \frac{-u_\theta}{r} = \frac{-\cos\theta}{r^2}$

$\therefore f'(r) = 0$ $\therefore f(r) = C$ $\therefore v = \frac{\cos\theta}{r} + C$

$\boxed{\frac{11}{50}}$

∵ $f(z) = u(r,\theta) + i v(r,\theta)$ analytic ∴ $r u_r = v_\theta$ & $u_\theta = -r v_r$

∴ $u_r + r u_{rr} = v_{\theta r}$ & $u_{\theta\theta} = -r v_{r\theta}$ (obtained by $\frac{\partial}{\partial r}$)

& $r u_{r\theta} = v_{\theta\theta}$ & $u_{\theta r} = -v_r - r v_{rr}$ (" " $\frac{\partial}{\partial \theta}$)

∴ $r^2 u_{rr} + r u_r + u_{\theta\theta} = (r v_{\theta r} - r u_r) + r u_r + (-r v_{r\theta}) = 0$

& $r^2 v_{rr} + r v_r + v_{\theta\theta} = (-r v_r - r u_{\theta r}) + r v_r + (r u_{r\theta}) = 0$

∴ Both u & v are harmonic

\# $h(x,y) = e^x \cos y$

∴ $h_x = e^x \cos y$ & $h_{xx} = e^x \cos y$

& $h_y = -e^x \sin y$ & $h_{yy} = -e^x \cos y$

∴ $h_{xx} + h_{yy} = e^x \cos y - e^x \cos y = 0$ ∴ h is harmonic in x, y

∵ $w = z^2 = x^2 - y^2 + i\, 2xy$ ∴ $u = x^2 - y^2$ & $v = 2xy$

∴ To prove that $h(x(u,v), y(u,v))$ is harmonic in u, v; we have to prove that $h_{uu} + h_{vv} = 0$. To do this, it is easier to consider $h_{xx} + h_{yy}$.

$h_x = h_u \cdot u_x + h_v \cdot v_x = h_u \cdot (2x) + h_v \cdot (2y)$

& $h_{xx} = h_{xu} \cdot u_x + h_{xv} \cdot v_x = [h_{uu}(2x) + h_{vu}(2y)](2x) + [h_{uv}(2x) + h_{vv}(2y)](2y)$

$= h_{uu}(4x^2) + h_{vv}(4y^2) + h_{uv}(8xy)$

& $h_y = h_u \cdot u_y + h_v \cdot v_y = h_u \cdot (-2y) + h_v (2x)$

& $h_{yy} = h_{yu} \cdot u_y + h_{yv} \cdot v_y = [h_{uu}(-2y) + h_{vu}(2x)][-2y] + [h_{uv}(-2y) + h_{vv}(2x)](2$

$= h_{uu}(4y^2) + h_{vv}(4x^2) + h_{uv}(-8xy)$

∴ $h_{xx} + h_{yy} = h_{uu}(4x^2 + 4y^2) + h_{vv}(4x^2 + 4y^2) + h_{uv}(8xy - 8xy) = (h_{uu} + h_{vv})(4x^2+$

∴ $h_{uu} + h_{vv} = (h_{xx} + h_{yy}) / [4(x^2 + y^2)] = 0$ ∴ h is harmonic in x, y & u, v

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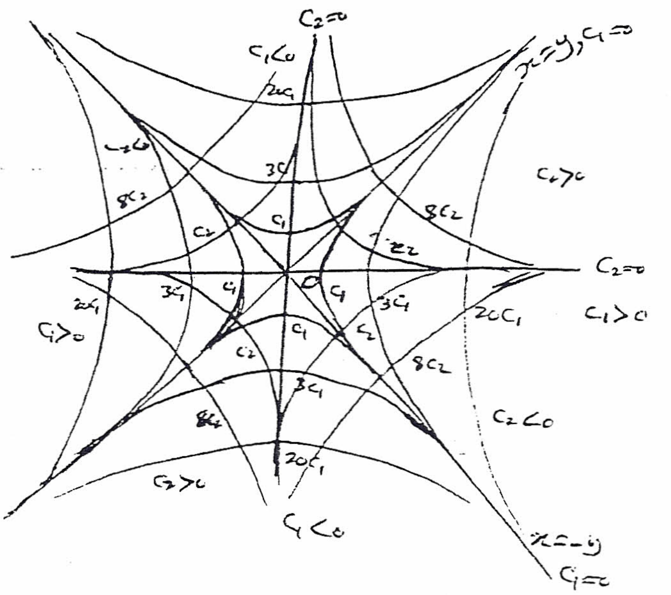
$\boxed{\frac{14}{51}}$ $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\,2xy = u + iv$

$\therefore$ $u = x^2 - y^2$ & $v = 2xy$

$\therefore$ $u = c_1 \Rightarrow x^2 - y^2 = c_1$ (1) & $v = c_2 \Rightarrow 2xy = c_2$ (2)

Eq. (1) is a hyperbola with assymptotes $x^2 - y^2 = 0$ or $x = \pm y$. When c_1 is positive then the hyperbola is along the x-axis and when c_1 is negative along y-axis when $c_1 = 0$ the hyperbola is just the assymptotes $x = \pm y$ as shown. Eq. (2) is a hyperbola rotated through angle $+\frac{\pi}{4}$ with assymptotes $2xy = 0$ or x & y axis. When c_2 is positive then the



hyperbola is in the first and third quadrants and when negative in t second and fourth quadrants, when $c_2 = 0$ the hyperbola is just the assymptotes $x = 0$ & $y = 0$ as shown above.

$\therefore$ The level curves are like the ones shown in Fig 17 - pp. 51

Check for orthogonality:

$y_1^2 = x^2 - c_1$ (from Eq. 1) , $y_2 = \frac{c_2}{2x}$ (from Eq. 2)

To find points of intersections we put $y_1 = y_2$

$\therefore$ Differentiating the above equations:

$\therefore$ $2y_1 y_1' = 2x$ & $y_2' = -\frac{c_2}{2x^2}$

$\therefore$ $2y_1 y_1' \cdot y_2' = \frac{-2x\,c_2}{2x^2} = \frac{-c_2}{x} \Rightarrow y_1' \cdot y_2' = \frac{-c_2}{2x\,y_1} = -\frac{y_2}{y_1} = \underset{\text{at inters}}{-1}$

$\therefore$ $y_1' \cdot y_2' = -1$ at points of intersections

$\therefore$ Tangents at points of intersections are normal to each other.

The curves corresponding to $c_1 = c_2 = 0$ (the assymptotes) inters at O but are not orthogonal because the condition for orthogonality (as set in Ex. 13) is that $f'(z) \neq 0$ i.e $2z \neq 0$ i.e $z \neq 0$

$\therefore$ The condition held for $z \neq 0$ and for $z = 0$ ($x = y = 0$) no orthogonality is guaranteed which agree to our findings.

—

$\boxed{\frac{1a}{55}}$ $\exp(2 \pm 3\pi i) = (\exp 2).\ \exp(\pm 3\pi i) = \exp 2 . \angle \pm 3\pi = -e^2$ ∴ OK

$\boxed{\frac{1b}{55}}$ ① $e^{\frac{2+\pi i}{4}} = e^{\frac{2}{4}+\frac{\pi i}{4}} = e^{\frac{1}{2}} . e^{\frac{i\pi}{4}} = \sqrt{e} . (\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}) = \sqrt{e} . \frac{1+i}{\sqrt{2}}$, ∴ OK

$\boxed{\frac{3a}{55}}$ $e^z = -2$ ∴ $z = \log(-2) = \log(2 \angle \pi + 2k\pi) = \ln 2 + i\pi(1+2k)$

k integer.

$\boxed{\frac{3b}{55}}$ ③ $e^z = 1 + \sqrt{3}\, i = 2 \angle \frac{\pi}{3} + 2k\pi = 2 e^{(\frac{\pi}{3}+2k\pi)i}$, k integer

∴ $z = \ln e^z = \ln(2 . e^{i(\frac{\pi}{3}+2k\pi)}) = \ln 2 + \ln e^{i(\frac{\pi}{3}+2k\pi)} = \ln 2 + i(\frac{\pi}{3} + 2k\pi)$

∴ solution is $z = \ln 2 + i(\frac{\pi}{3} + 2k\pi)$, k integer.

$\boxed{\frac{3c}{55}}$ ② $e^{2z-1} = 1 = 1 \angle 2n\pi = e^{i 2n\pi}$ ∴ $2z - 1 = i\, 2n\pi$

∴ $z = \frac{1}{2} + n\pi i$, n integer.

$\boxed{\frac{12}{56}}$ ③ $f(z) = e^{\bar{z}} = e^{x-iy} = e^x . e^{-iy} = e^x(\cos y - i \sin y) =$

$= e^x \cos y - i e^x \sin y = u + iv$

∴ $u = e^x \cos y$ & $v = -e^x \sin y$

∴ $u_x = e^x \cos y$ & $v_y = -e^x \cos y$, ∴ $u_x = v_y$ only when $y = (2n+1)\frac{\pi}{2}$

& $u_y = -e^x \sin y$ & $v_x = -e^x \sin y$, ∴ $u_y = -v_x$ only when $y = n\pi$

∴ For CRE to hold conditions are contradicting

∴ CRE is not satisfied anywhere

∴ $f(z) = e^{\bar{z}}$ is not differentiable anywhere and hence not analytic.

$\boxed{\frac{16}{56}}$ (2) $f(z) = u + iv$ ∵ f is analytic ∴ $u_x = v_y$ & $u_y = -v_x$

$U = e^u \cos v = \mathrm{Re}\, e^f$, $V = e^u \sin v = \mathrm{Im}\, e^f$

∴ $U + iV = F = e^f$ ∵ e^z & $f(z)$ are analytic ∴ Composition is analytic.

∴ $e^f = F$ is analytic because it is composed of analytic functions.

OR $U_x = (e^u \cos v) u_x + e^u(-\sin v) v_x$, $V_y = (e^u \sin v) \cdot u_y + (e^u \cos v) \cdot v_y =$

$=$ (using $u_x = v_y$ & $u_y = -v_x$) $e^u \sin v\,(-v_x) + e^u \cos v \cdot u_x = U_x$ ∴ OK

& $U_y = e^u \cos v \cdot u_y + e^u(-\sin v) v_y$, $V_x = e^u \sin v \cdot u_x + e^u \cos v \cdot v_x =$

$= e^u \sin v \cdot v_y - e^u \cos v \cdot u_y = -U_y$ ∴ OK ∴ $F = U + iV$ is analytic because it satisfy CRE's. ∴ Both U & V are harmonic functions and additionally V is a harmonic conjugate of U.

$\boxed{\frac{6}{59}}$ (4) $|\sin z|^2 = \sin^2 x + \sinh^2 y \geq \sin^2 x = |\sin x|^2$

∴ $|\sin z| \geq |\sin x|$ ∴ OK

& $|\cos z|^2 = \cos^2 x + \sinh^2 y \geq \cos^2 x = |\cos x|^2$

∴ $|\cos z| \geq |\cos x|$ ∴ OK.

$\boxed{\frac{10a}{59}}$ $\cos(i\bar{z}) = \cos[i(x - iy)] = \cos(y + ix) = \cos y \cosh x - i \sin y \sinh x$

$\overline{\cos iz} = \overline{\cos i(x+iy)} = \overline{\cos(-y+ix)} = \overline{\cos y \cosh x - i \sin(-y) \sinh x} =$

$= \overline{\cos y \cosh x + i \sin y \sinh x} = \cos y \cosh x - i \sin y \sinh x$

∴ $\cos(i\bar{z}) = \overline{\cos iz}$ for all z ∴ OK

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$LHS = \sin(i\bar{z}) = \sin\left(i\,\overline{(x+iy)}\right) = \sin\left(i(x-iy)\right) = \sin(ix+y) =$

$= \sin y \cosh x + i\cos y \sinh x$

$RHS = \overline{\sin iz} = \overline{\sin i(x+iy)} = \overline{\sin(ix-y)} = \overline{-\sin y\cosh x + i\cos y \sinh x}$

$= -\sin y \cosh x - i\cos y \sinh x$

$\because LHS = RHS$

$\therefore \sin y \cosh x = -\sin y \cosh x \Rightarrow 2\sin y \cosh x = 0 \quad (1)$

& $\cos y \sinh x = -\cos y \sinh x \Rightarrow 2\cos y \sinh x = 0 \quad (2)$

solution of (1) is $y = k\pi$ only, where $k = 0, \pm1, \pm2, \ldots$

so, for (2) to be true:

$\therefore 2\cos k\pi \cdot \sinh x = 0 \Rightarrow 2(-1)^k \cdot \sinh x = 0 \Rightarrow \sinh x = 0 \Rightarrow x = 0$

$\therefore$ LHS = RHS only when $z = x + iy = 0 + ik\pi$

$\therefore$ $z = ik\pi$ is the solution to the equation and it hold OK for it.

(check:

$LHS = \sin i\bar{z} = \sin i(\overline{ik\pi}) = \sin i(-ik\pi) = \sin k\pi = 0$

$RHS = \overline{\sin(iz)} = \overline{\sin i(ik\pi)} = \overline{\sin -k\pi} = \overline{-\sin k\pi} = -\sin k\pi = 0 \quad \therefore OK$)

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$\sin z = \cosh 4$

$\therefore \sin x \cosh y + i\cos x \sinh y = \cosh 4$

$\therefore \sin x \cosh y = \cosh 4 \quad (1)$ & $\cos x \sinh y = 0 \quad (2)$

From (2) $\therefore \sinh y = 0 \Rightarrow y = 0 \ (3)$ OR $\cos x = 0 \ \therefore x = \text{odd}\frac{\pi}{2} \ (4)$

(3) in (1) $\therefore \sin x \cosh 0 = \cosh 4 \ \therefore \sin x = \cosh 4 > 1 \ \therefore \times$

(4) in (1) $\therefore \sin(\text{odd}\frac{\pi}{2}) \cosh y = \cosh 4$

$\therefore$ either $\sin(n2\pi + \frac{\pi}{2})\cosh y = \cosh 4 \Rightarrow \cosh y = \cosh 4 \ \therefore y = \pm 4$

OR $\sin(2n\pi - \frac{\pi}{2})\cosh y = \cosh 4 \Rightarrow -\cosh y = \cosh 4 \ \therefore \times$

$\therefore$ The solution is when $z = x + iy$ where $x = 2n\pi + \frac{\pi}{2}, y = \pm 4$

$\therefore z = 2n\pi + \frac{\pi}{2} \pm i4 = \frac{\pi}{2}(4n+1) \pm i4$, n integer

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another way:

$\sin z = \sin\left[(z - \frac{\pi}{2} - 2n\pi) + \frac{\pi}{2}\right] = \cos\left(\left[(z - \frac{\pi}{2} - 2n\pi)/i\right] * i\right) = \cosh\left[\frac{(z - \pi/2 - 2n\pi)}{i}\right] = \cosh 4$

$\boxed{\frac{14}{59}}$ (4) $\cos z = 2$ $\therefore$ $\cos x \cosh y - i \sin x \sinh y = 2 + i0$

$\therefore$ Re = Re & Im = Im

$\therefore$ $\cos x \cosh y = 2$ (1) & $\sin x \sinh y = 0$ (2)

$\therefore$ From (2) OR

- $\sin x = 0 \Rightarrow x = m\pi$ (m integer) $\therefore (1) \Rightarrow \cosh y = 2$ (m even) $\therefore y = \cosh^{-1} 2$ or $\cosh y = -2$ (m odd) $\therefore y \in \{\phi\}$
- $\sinh y = 0 \Rightarrow y = 0$ $\therefore (1) \Rightarrow \cos x = 2$ $\therefore x \in \{\phi\}$

$\therefore$ The only solution is $x = m\pi$, m even integer & $y = \cosh^{-1} 2$ (or $x = 2n\pi$, n integer)

((Note: if $\cosh y = x$ $\therefore$ $x = \frac{e^y + e^{-y}}{2} \Rightarrow e^{2y} - 2x e^y + 1 = 0 \Rightarrow$

$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1} \Rightarrow y = \ln\left(x \pm \sqrt{x^2-1}\right) = \cosh^{-1} x$

$\therefore$ $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$ or $\ln\left(x - \sqrt{x^2-1}\right) = \ln\left[\frac{\left(x-\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right)}{\left(x+\sqrt{x^2-1}\right)}\right] =$

$= \ln\left[\frac{x^2 - (x^2-1)}{x+\sqrt{x^2-1}}\right] = \ln\left[\frac{1}{x+\sqrt{x^2-1}}\right] = \cancel{\ln 1}^{0} - \ln\left(x+\sqrt{x^2-1}\right) = -\ln\left(x+\sqrt{x^2-1}\right)$

$\therefore$ $\cosh^{-1} x = \pm \ln\left(x + \sqrt{x^2-1}\right)$))

$\therefore$ The solution is $z = x + iy = 2n\pi + i\cosh^{-1} 2 = 2n\pi \pm i \ln(2+\sqrt{3})$

$\boxed{\frac{17a}{59}}$ (3) $\sin \bar{z} = \sin x \cosh(-y) + i \cos x \sinh(-y) = \sin x \cosh y - i \cos x \sinh y =$

$= u + iv$

$\therefore$ $u = \sin x \cosh y$ & $v = -\cos x \sinh y$

$\therefore$ $u_x = \cos x \cosh y$ & $v_y = -\cos x \cosh y$ $\therefore$ $u_x = v_y$ only when $x = \frac{(2n+1)\pi}{2}$, n integer

but then $u_y = \sin x \sinh y = (-1)^n \sinh y$ & $v_x = \sin x \sinh y = (-1)^n \sinh y$

$\therefore$ $u_y = -v_x$ only when $\sinh y = 0 \Rightarrow y = 0$

$\therefore$ CRE are OK at the point $\left[\left(\frac{2n+1}{2}\right)\pi, 0\right]$ only $\therefore$ $\sin \bar{z}$ is not analytic anywhere.

$\boxed{\frac{176}{59}}$ (2) $\cos \bar{z} = \cos(x - iy) = \cos x \cos iy + \sin x \sin iy =$

$= \cos x \cosh y + i \sin x \sinh y = u + iv$

$\therefore$ $u = \cos x \cosh y$ & $v = \sin x \sinh y$

$\therefore$ $u_x = -\sin x \cosh y$ & $v_y = \sin x \cosh y$

$\therefore$ $u_x = v_y \Rightarrow -\sin x \cosh y = \sin x \cosh y \Rightarrow 2 \sin x \cosh y = 0$

$\cosh y \neq 0$ $\therefore \sin x = 0$ $\therefore x = m\pi$, m integer for $u_x = v_y$

$u_y = \cos x \sinh y$ & $v_x = \cos x \sinh y$

$\therefore$ If $u_y = -v_x$ then $\cos x \sinh y = -\cos x \sinh y \Rightarrow 2 \cos x \sinh y = 0$

$\therefore$ either $\cos x = 0 \Rightarrow x = \frac{2n+1}{2}\pi$ (rejected)

or $\sinh y = 0 \Rightarrow y = 0$

$\therefore$ For Cauchy to be happy $x = m\pi$ & $y = 0$

$\therefore$ The function $\cos \bar{z}$ is differentiable on $(m\pi + i0)$ and nowhere else $\therefore$ The function is analytic nowhere because (for analiticity at z_0, the function must be differentiable at z_0 and its ϵ-neighbourhood which is not the case.

$\therefore$ $\cos \bar{z}$ is analytic nowhere.

$\frac{4}{61}$

$$\cosh z = \frac{e^{z}+e^{-z}}{2} = (e^{x+iy}+e^{-x-iy})/2 =$$

$$= \frac{1}{2}\left(e^{x+iy}+e^{x-iy}-e^{x-iy}+e^{-x-iy}\right) =$$

$$= \frac{1}{2}\left[e^{x}\cdot(e^{iy}+e^{-iy}) - e^{-iy}(e^{x}-e^{-x})\right] =$$

$$= e^{x}\cdot\frac{\cos y + i\sin y + \cos y - i\sin y}{2} - (\cos y - i\sin y)\frac{e^{x}-e^{-x}}{2} =$$

$$= e^{x}\cos y - \cos y\cdot\frac{e^{x}-e^{-x}}{2} + i\sin y\,\sinh x =$$

$$= \cos y\cdot\frac{2e^{x}-e^{x}+e^{-x}}{2} + i\sin y\,\sinh x = \cos y\cdot\frac{e^{x}+e^{-x}}{2} + i\sin y\,\sinh x$$

$$= \cos y\cdot\cosh x + i\sin y\cdot\sinh x$$ (You can use Eq (12) straight)

$\therefore$ $\cosh z = \cosh x\cos y + i\sinh x\sin y$

$$\therefore\ |\cosh z|^2 = \cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y =$$
$$= (1+\sinh^2 x)\cos^2 y + \sinh^2 x\sin^2 y =$$
$$= \cos^2 y + \sinh^2 x(\cos^2 y + \sin^2 y) = \cos^2 y + \sinh^2 x$$

$\therefore$ $|\cosh z|^2 = \sinh^2 x + \cos^2 y$, $\therefore$ OK.

$\therefore$ $|\cosh z|^2 \geq \sinh^2 x = |\sinh x|^2$ $\therefore$ $|\cosh z| \geq |\sinh x|$ (1)

$\therefore$ $|\cosh z|^2 = \sinh^2 x + \cos^2 y$ $\therefore$ $|\cosh z|^2 = (\cosh^2 x - 1) + \cos^2 y$

$\therefore$ $\cosh^2 x = |\cosh z|^2 + 1 - \cos^2 y = |\cosh z|^2 + \sin^2 y \geq |\cosh z|^2$

$\therefore$ $\cosh x \geq |\cosh z|$ (2)

$\therefore$ From (1) & (2) $\Rightarrow$ $|\sinh x| \leq |\cosh z| \leq \cosh x$, OK

$\frac{8}{61}$ Zeros & Singularities of $\tanh z$

$\therefore f(z) = \dfrac{\sinh z}{\cosh z}$

zeros at $\sinh z = 0 \Rightarrow \sinh i\left(\frac{z}{i}\right) = 0 \Rightarrow i \sin \frac{z}{i} = 0 \Rightarrow \frac{z}{i} = n\pi$

$\therefore$ Zeros of f are at $z = n\pi i$, n integer

Singularities at $\cosh z = 0 \Rightarrow \cosh i\left(\frac{z}{i}\right) = 0 \Rightarrow \cos\frac{z}{i} = 0 \Rightarrow \frac{z}{i} = \text{odd } \frac{\pi}{2}$

$\therefore$ Singularities of f are at $z = (2n+1)\pi/2$, n integer.

$\frac{9b}{61}$ (2) $\sinh z = i \Rightarrow \sinh\left(\frac{iz}{i}\right) = i \Rightarrow \sinh i\left(\frac{z}{i}\right) = i$

$\therefore i \sin\left(\frac{z}{i}\right) = i$

$\therefore \sin\left(\frac{x+iy}{i}\right) = 1 \Rightarrow \sin\frac{x}{i}\cos y + \cos\frac{x}{i}\sin y = 1$

$\therefore \sin(-ix)\cos y + \cos(-ix)\sin y = 1$

$\therefore -\sin ix \cos y + \cos ix \sin y = 1$

$\therefore -i\sinh x \cos y + \cosh x \sin y = 1$ (multiply by i)

$\therefore \sinh x \cos y + i\cosh x \sin y = i$

$\therefore \sinh x \cos y = 0 \Rightarrow$ either $x = 0$ or $y = \frac{2n+1}{2}\pi$

& $\cosh x \sin y = 1$

$x=0 \Rightarrow \cosh 0 \sin y = 1 \Rightarrow \sin y = 1 \quad \therefore y = \frac{4m+1}{2}\pi$

$y = \frac{2n+1}{2}\pi \Rightarrow \cosh x \sin\frac{2n+1}{2}\pi = 1$

n even: $\cosh x = 1 \quad \therefore x = 0$ & $y = \frac{4m+1}{2}\pi$

n odd: $\cosh x = -1$ ✗

$\therefore$ The solution is at $x = 0$, $y = \frac{4m+1}{2}\pi = \frac{\pi}{2} + 2m\pi$

$\therefore$ The roots are $z = \left(2m + \frac{1}{2}\right)i\pi$, m integer.

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$\cosh z = -2$

$\therefore \cosh x \cos y + i \sinh x \sin y = -2$

$\therefore \cosh x \cos y = -2 \quad (1) \quad \& \quad \sinh x \sin y = 0 \quad (2)$

From (2) $\therefore y = k\pi \quad (3) \quad$ or $\quad x = 0 \quad (4)$

(3) $\rightarrow$ (1) $\therefore \cosh x \cos k\pi = -2$

$\therefore$ either k even $\therefore \cosh x \cos 2n\pi = -2 \quad \therefore \cosh x = -2 \quad \therefore \times$

or k odd $\therefore \cosh x \cos(2n+1)\pi = -2 \quad \therefore \cosh x = 2 \quad \boxed{*}$

$\therefore \dfrac{e^{x} + e^{-x}}{2} = 2 \quad \therefore e^{2x} - 4e^{x} + 1 = 0 \quad$ Let $e^{x} = t$

$\therefore t^{2} - 4t + 1 = 0 \quad \therefore t = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\therefore t = 2 \pm \sqrt{3} \quad \therefore e^{x} = 2 \pm \sqrt{3} \quad \therefore x = \ln(2 \pm \sqrt{3})$

$\therefore x = \ln(2+\sqrt{3})$ or $x = \ln(2-\sqrt{3}) = \ln\left[\dfrac{(2-\sqrt{3})(2+\sqrt{3})}{2+\sqrt{3}}\right] =$

$= \ln\left[\dfrac{4-3}{2+\sqrt{3}}\right] = \ln \dfrac{1}{2+\sqrt{3}} = -\ln(2+\sqrt{3})$

$\therefore x = \pm \ln(2+\sqrt{3})$

(4) $\rightarrow$ (1) $\therefore \cosh 0 \cos y = -2 \quad \therefore \cos y = -2 < -1 \quad \therefore \times$

$\therefore$ The solution is $z = x + iy = \pm \ln(2+\sqrt{3}) + (2n+1)\pi i$

n integer

\# another way:

$\cosh z = \cosh\left[\left(\frac{z}{i}\right) i\right] = \cos\left[\frac{z}{i} + 2k\pi + \pi + \pi\right] = -\cos\left(\frac{z}{i} + (2k+1)\pi\right) = -2$

$\therefore \cos\left[(z + (2k+1)\pi i)\, i\right] = 2 \quad \therefore \cosh\left[z + (2k+1)\pi i\right] = \cosh \cosh^{-1} 2$

$\therefore z + (2k+1)\pi i = \cosh^{-1} 2 =$ (from $\boxed{*}$ above) $\pm \ln(2+\sqrt{3})$

$\therefore z = \pm \ln(2+\sqrt{3}) - (2k+1)\pi i = \underline{\pm \ln(2+\sqrt{3}) + \text{odd } \pi i}$ ($\therefore$ OK)

$\boxed{\frac{1.19}{57}}$ $\mathrm{Log}(-ei) = \log(e\angle -\frac{\pi}{2}) = \ln e + i(-\frac{\pi}{2}) = 1 - \pi i/2 \quad \therefore \text{OK}$

$\boxed{\frac{2c}{67}}$ $\log i = (2n+\frac{1}{2})\pi i$

$\text{LHS} = \log i = \log(1\angle \frac{\pi}{2}+2k\pi) = \ln 1 + i\pi(\frac{1}{2}+2k) = 0 + (2k+\frac{1}{2})\pi i =$
$= \text{RHS} \quad \therefore \text{OK}$

$\boxed{\frac{2d}{67}}$ ② $\log(i^{\frac{1}{2}}) = \frac{1}{2}\log i = \frac{1}{2}\log[1\angle \frac{\pi}{2}+2k\pi] = \frac{1}{2}[\ln 1 + i(\frac{\pi}{2}+2k\pi)] =$

$= \frac{1}{2}[0 + i(\frac{\pi}{2}+2k\pi)] = i(\frac{\pi}{4}+k\pi) = (k+\frac{1}{4})\pi i$, k integer, $\therefore$ OK

$\boxed{\frac{3}{67}}$ $\log z = (\pi/2)i \quad \therefore \exp(\log z) = \exp(i\pi/2) \quad \therefore z = e^{i\pi/2} = i$

$\boxed{\frac{9}{67}}$ ⓐ ② If $z \in \{r>0,\ \theta\in(\frac{\pi}{4}, \frac{9\pi}{4})\}$ $\quad \therefore \log(i^2) = \log(-1) = \log(1\angle\pi) =$

$= \ln 1 + i\pi = 0 + i\pi = i\pi$

& $2\log i = 2\log(1\angle\frac{\pi}{2}) = 2(\ln 1 + i\frac{\pi}{2}) = 2(0+\frac{i\pi}{2}) = i\pi$

$\therefore \log i^2 = 2\log i$, in the above branch of z.

ⓑ ② If $z \in \{r>0,\ \theta\in(\frac{3\pi}{4}, \frac{11\pi}{4})\}$ $\quad \therefore \log(i^2) = \log(-1) = \log(1\angle\pi) =$

$= \ln 1 + i\pi = 0 + i\pi = i\pi$

& $2\log i = 2\log(1\angle\frac{5\pi}{2}) = 2(\ln 1 + i\frac{5\pi}{2}) = 2(0 + i\frac{5\pi}{2}) = i5\pi$

$\therefore \log i^2 \neq 2\log i$, in the above branch of z.

$\frac{1a}{71}$

$$(1+i)^i = \exp\left[\log\left((1+i)^i\right)\right] = \exp\left[i \log(1+i)\right] =$$

$$= \exp\left[i \cdot \log\left(\sqrt{2}\,\angle\tfrac{\pi}{4}+2k\pi\right)\right] = \exp\left[i\cdot\left(\ln\sqrt{2} + i\left(\tfrac{\pi}{4}+2k\pi\right)\right)\right] =$$

$$= \exp\left[i \ln(2^{\frac{1}{2}}) - \tfrac{\pi}{4}(1+8k)\right] = \exp\left[\tfrac{i}{2}\ln 2\right]\cdot\exp\left[-\tfrac{\pi}{4}+2n\pi\right]$$

$n=-k$

$$= e^{(2n-\frac{1}{4})\pi}\,\angle\tfrac{\ln 2}{2} = e^{(2n-.25)\pi}\,\angle .3466\text{ rad}.$$

n, k are integers ($\doteq$ OK)

$\frac{1b}{71}$ ② Let $w = (-1)^{1/\pi}$ $\therefore \log w = \frac{1}{\pi}\log(-1)$ $\therefore w = \exp\left(\frac{1}{\pi}\log(-1)\right) =$

$= \exp\left(\frac{1}{\pi}\log\left(1\,\angle\pi+2k\pi\right)\right) = \exp\left(\frac{1}{\pi}\left(\ln 1 + i(\pi+2k\pi)\right)\right) = \exp\left(\frac{1}{\pi}\left(0+i\pi(1+2k)\right)\right)$

$= \exp\left(\frac{1}{\pi}\cdot i\pi(1+2k)\right) = \exp\left[(1+2k)i\right]$, k integer

$\therefore (-1)^{1/\pi} = \angle 1+2k$, k integer.

$\frac{2a}{71}$ ② Let $z = i^i$

$\therefore \operatorname{Log} z = i \operatorname{Log} i = i \operatorname{Log}\left(1\,\angle\frac{\pi}{2}\right) = i\left(\ln 1 + i\frac{\pi}{2}\right) = i\left(0+\frac{i\pi}{2}\right) = -\frac{\pi}{2}$

$\therefore z = e^{-\pi/2}$

$\therefore$ Principal value of i^i is $e^{-\pi/2}$

Let $z = i^{(1+i)}$ $\therefore \operatorname{Log} z = (1+i)\operatorname{Log} i = (1+i)\operatorname{Log}\left(1\,\angle\pi/2\right) =$

$= (1+i)\left(\ln 1 + i\frac{\pi}{2}\right) = (1+i)\left(0+i\pi/2\right) = \frac{i\pi}{2} - \frac{\pi}{2}$

$\therefore z = e^{\operatorname{Log} z} = e^{-\frac{\pi}{2}+\frac{i\pi}{2}} = e^{-\pi/2}\cdot e^{i\pi/2} = e^{-\pi/2}\left(\cancel{\cos\pi/2}^{\,0} + i\sin\pi/2\right) = i e^{-\pi/2}$

$\therefore$ Principal value of i^{1+i} is $i e^{-\pi/2}$.

$\frac{2b}{71}$ ② Let $w = \left[\frac{e}{2}\cdot(-1-i\sqrt{3})\right]^{3\pi i} \Rightarrow \log w = 3\pi i \log\left[\frac{e}{2}\cdot(-1-i\sqrt{3})\right] \Rightarrow$

$\therefore w = \exp\left[3\pi i \log\{\frac{e}{2}(-1-i\sqrt{3})\}\right] = \exp\left[3\pi i \log\left(e\,\angle-\frac{2\pi}{3}\right)\right] = \exp\left[3\pi i\left(\ln e - \frac{2\pi}{3}i\right)\right] =$

$= \exp\left[3\pi i\left(1-\frac{2\pi i}{3}\right)\right] = \exp\left[3\pi i + 2\pi^2\right] = \left[\exp(2\pi^2)\right]\cdot\left[\exp(3\pi i)\right] = -\exp(2\pi^2)$

$\therefore$ Principal branch of w evaluate to $\left[-\exp(2\pi^2)\right]$.

$\boxed{\frac{7c}{71}}$ ② $(1-i)^{4i} = w$

$\therefore \operatorname{Log} w = 4i \operatorname{Log}(1-i) = 4i \operatorname{Log}(\sqrt{2}\angle{-\pi/4}) = 4i(\ln\sqrt{2} - i\pi/4) =$

$= (4\ln\sqrt{2})i + \pi = \pi + i\ln(\sqrt{2})^4 = \pi + i\ln 4$

$\therefore w = e^{\pi + i\ln 4} = e^{\pi}\angle{\ln 4} = e^{\pi}(\cos\ln 4 + i\sin\ln 4)$

$\therefore (1-i)^{4i} = e^{\pi}\cdot(\cos\ln 4 + i\sin\ln 4)$. (principal value)

$\boxed{\frac{6}{71}}$ ② $\frac{d}{dz} c^z = c^z \log c$

Let $w = c^z$ $\therefore \log w = z\log c$

$\therefore w = \exp(z\log c)$

$\therefore \frac{dw}{dz} = [\exp(z\log c)] \cdot \frac{d}{dz}(z\log c) = w\cdot\log c$

$\therefore \frac{dc^z}{dz} = c^z\log c$, $\therefore$ OK.

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$$\tan^{-1}(1+i)=\frac{1}{2i}\log\left(\frac{1+i-1}{1-i+1}\right)=\frac{1}{2i}\log\left(\frac{2+i}{5}\,i\right)=\frac{1}{2i}\left[\log\frac{\angle\frac{\pi}{2}+\tan^{-1}\frac{1}{2}+2k\pi}{\sqrt{5}}\right]$$

$$=\frac{1}{2i}\log\left(\frac{1}{\sqrt{5}}\angle\frac{\pi}{2}+\frac{\pi}{2}-\tan^{-1}2+2k\pi\right)=\frac{1}{2i}\left(\ln 5^{-\frac{1}{2}}+i(\pi+2k\pi-\tan^{-1}2)\right)=$$

$$=\frac{1}{2i}\left(-\frac{1}{2}\ln 5+i\left(\pi(1+2k)-\tan^{-1}2\right)\right)=\frac{\pi(1+2k)-\tan^{-1}2}{2}+i\frac{\ln 5}{4}.$$

\# Let $\tan^{-1}(w)=z$ $\quad\therefore\quad w=\tan z$

$\therefore (w)^2=\tan^2 z=\sec^2 z-1 \quad \therefore \sec^2 z=1+(\tan z)^2=1+w^2$

$\therefore \cos^2 z=\dfrac{1}{1+(w)^2} \quad \therefore \cos z=(1+w^2)^{-1/2}=\cos\left[i\left(\frac{z}{i}\right)\right]$

$\therefore \cosh\left(\frac{z}{i}\right)=(1+w^2)^{-1/2}=\left(e^{z/i}+e^{-z/i}\right)/2 \qquad$ let $e^{z/i}=t$

$\therefore (1+w^2)^{-1/2}=\dfrac{t+t^{-1}}{2}=\dfrac{t^2+1}{2t} \quad\therefore\quad t^2-2t(1+w^2)^{-1/2}+1=0$

$\therefore t=\dfrac{2(1+w^2)^{-1/2}\pm\sqrt{4(1+w^2)^{-1}-4}}{2}=\dfrac{1\pm iw}{\sqrt{1+w^2}}=e^{z/i}$

$\therefore z=i\left[\log(1\pm iw)-\frac{1}{2}\log(1-i^2w^2)\right]=\frac{i}{2}\left[2\log(1\pm iw)-\log(1+iw)-\log(1-iw)\right]$

$=\frac{i}{2}(\pm)\left[\log(1+iw)-\log(1-iw)\right]=\pm\frac{i}{2}\log\left(\frac{1+iw}{1-iw}\right)=\pm\frac{1}{2i}\log\left(\frac{1+iw}{1-iw}\right)$

(Note: when w is real then $\log\left(\frac{1+iw}{1-iw}\right)=\ln\left|\frac{1+w^2}{1+w^2}\right|+2i\tan^{-1}w=2i\tan^{-1}w \quad\therefore z=\pm\tan^{-1}w$) $-$ to rejected

$\therefore z=\tan^{-1}w=\frac{1}{2i}\log\left(\frac{1+iw}{1-iw}\right)\therefore \tan^{-1}(1+i)=\frac{1}{2i}\log\left(\frac{1+i-1}{1-i+1}\right)=\frac{1}{2i}\log\left(\frac{2+i}{5}i\right)=\frac{1}{2i}\left[-\frac{1}{2}\ln 5+i(\text{odd}\,\pi-\tan^{-1}2)\right]=\frac{(2n+1)\pi-\tan^{-1}2}{2}+\frac{i\ln 5}{4}$

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③ Let $\cosh^{-1}(-1)=z$

$\therefore -1=\cosh z=\cosh x\cos y+i\sinh x\sin y$

$\therefore$ Re = Re $\quad\therefore \cosh x\cos y=-1 \qquad (1)$

& Im = Im $\quad\therefore 0=\sinh x\sin y \quad\therefore y=k\pi$ or $x=0$ into (2)

$\therefore$ when $y=k\pi \Rightarrow \cosh x\cdot\cos k\pi=-1 \Rightarrow$ k odd $\cosh x=1 \therefore x=0$; even $\cosh x=-1 \therefore \times$ (rejected)

& when $x=0 \Rightarrow \cosh 0\cdot\cos y=-1 \therefore \cos y=-1 \therefore y=(2n+1)\pi$

$\therefore$ The solutions are $y=k\pi$, k odd & $x=0$, OR $y=(2n+1)\pi$, $x=0$

$\therefore z=0+(2n+1)\pi i$

$\therefore \cosh^{-1}(-1)=(2n+1)\pi i$

8d/71 Let $\tanh^{-1} 0 = z$ $\quad\therefore\ 0 = \tanh z = \dfrac{\sinh z}{\cosh z}$

$\therefore\ \sinh z = 0 \quad \therefore\ \sinh\left[i\left(\frac{z}{i}\right)\right] = 0 \quad \therefore\ i \sin\left(\frac{z}{i}\right) = 0 \quad \therefore\ \frac{z}{i} = k\pi,\ k = 0, 1, \ldots$

$\therefore\ z = k\pi i$ is the solution to $\tanh^{-1} 0$.

10a/71 $\because\ \sin z = 2 \quad \therefore\ \sin x \cosh y + i \cos x \sinh y = 2 + i0$

$\therefore\ \cos x \sinh y = 0 \implies y = 0$ or $x = \text{odd}\ \pi/2$

& $\sin x \cosh y = 2$

$\therefore$ If $y = 0 \implies \sin x . \cosh 0 = 2 \implies \sin x = 2$, impossible

or $x = \text{odd}\ \pi/2 \implies \sin(\text{odd}\ \frac{\pi}{2}) \cosh y = 2$

$\because$ odd $\pi/2$ is $\frac{\pi}{2} + 2k\pi$ or $-\pi/2 + 2k\pi$

$\therefore\ \sin\left(\frac{\pi}{2} + 2k\pi\right) \cosh y = 2 \implies \cosh y = 2 \implies y = \pm\cosh^{-1} 2$

or $\sin\left(-\frac{\pi}{2} + 2k\pi\right) \cosh y = 2 \implies -\cosh y = 2 \implies$ no solution

$\therefore$ The only solution is $x = \frac{\pi}{2} + 2k\pi$, $y = \pm\cosh^{-1} 2$

$\therefore\ z = (1 + 4k)\frac{\pi}{2} \pm i \cosh^{-1} 2$

\# $\sin z = i \implies \sin(x + iy) = i \implies \sin x \cos iy + \cos x \sin iy = i$

$\therefore\ \sin x \cosh y + i \cos x \sinh y = 0 + i$

$\therefore\ \sin x \cosh y = 0 \implies \sin x = 0 \implies x = m\pi$

& $\cos x \sinh y = 1 \implies \cos m\pi \sinh y = 1$ $\begin{cases} m\text{ odd} \implies -\sinh y = 1 \quad \therefore\ y = \sinh^{-1}(-1) \\ m\text{ even} \implies \sinh y = 1 \quad \therefore\ y = \sinh^{-1} 1 \end{cases}$

$\therefore$ The roots are $2n\pi + i \sinh^{-1} 1$ & $(2n+1)\pi - i \sinh^{-1} 1$

(Note: $\sinh y$ is odd function and so is $\sinh^{-1} y \implies \sinh(-y) = -\sinh y$ & $\sinh^{-1}(-y) = -\sinh^{-1} y$, check: $\sinh^{-1} y = x \implies \sinh x = y = \dfrac{e^x - e^{-x}}{2} \implies$

$e^x - e^{-x} = 2y \implies e^{2x} - 2y e^x - 1 = 0 \implies e^x = \dfrac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1} \implies$

$\therefore\ x = \ln(y \pm \sqrt{y^2+1}) = \begin{cases} \ln(y + \sqrt{y^2+1}) \\ \ln(-\text{ve})\ \text{rejected} \end{cases} = \ln(y + \sqrt{y^2+1}) = \sinh^{-1} y$ 36

$\therefore\ \sinh^{-1}(-y) = \ln(-y + \sqrt{y^2+1}) = \ln\left[(-y + \sqrt{y^2+1})\dfrac{y + \sqrt{y^2+1}}{y + \sqrt{y^2+1}}\right] = \ln \dfrac{-y^2 + y^2 + 1}{y + \sqrt{y^2+1}} = \ln \dfrac{1}{y + \sqrt{y^2+1}} = -\ln(y + \sqrt{y^2+1})$ $= -\sinh^{-1} y$

$\frac{2}{77}$ ① Lets find the immage of (x, y) first.

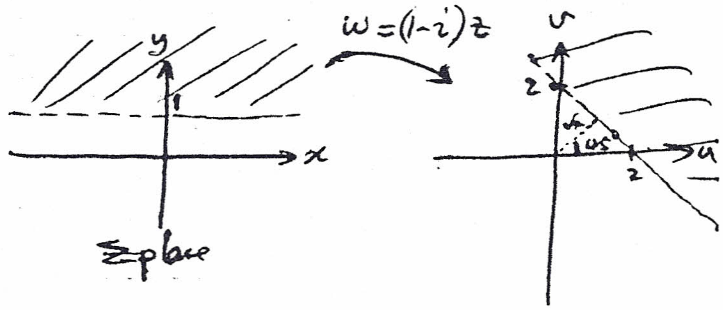
$\therefore\; w = iz + i = i(x+iy) + i = i(\cancel{x}+iy) + i = i^2 y + i(1+x) =$
$= -y + i(1+x) = u + iv$

$\therefore\; u = -y \;\;\&\;\; v = 1 + x$

$\therefore$ The point $(0, y)$ maps onto $(-y, 1)$
$\therefore$ The line $x = 0$ maps onto the line $v = 1$
$\therefore$ The region $x \geqslant 0$ maps onto the region $v \geqslant 1$
$\therefore$ The half plane $x > 0$ maps onto the half plane $v > 1$ $\therefore$ OK

$\frac{4}{77}$ ② $w = (1-i)z = \sqrt{2}\angle{-45^\circ} * z$

$\therefore$ It is magnification by $\sqrt{2}$
& rotation by $-45°$



OR, $\because\; z = x + iy$

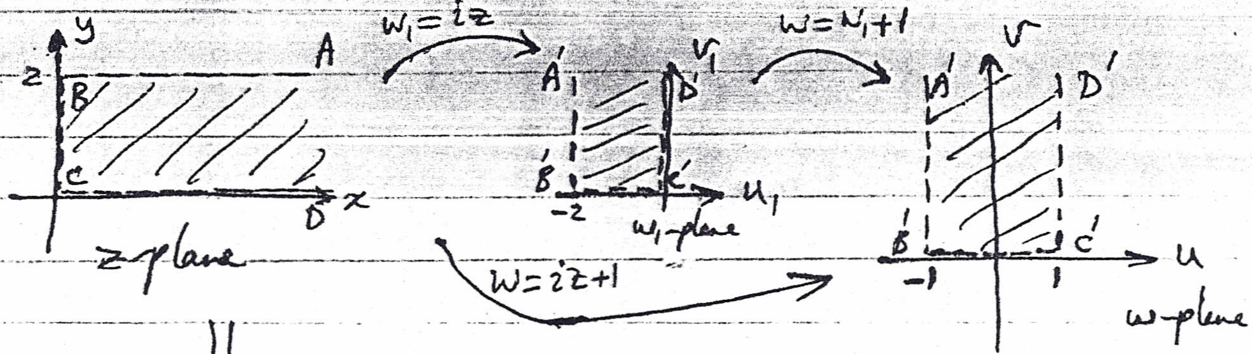
$\therefore\; w = (1-i)z = (1-i)(x+iy)$
$= x + y + i(y - x) = u + iv$

$\therefore\; u = x + y \;\;\&\;\; v = y - x$

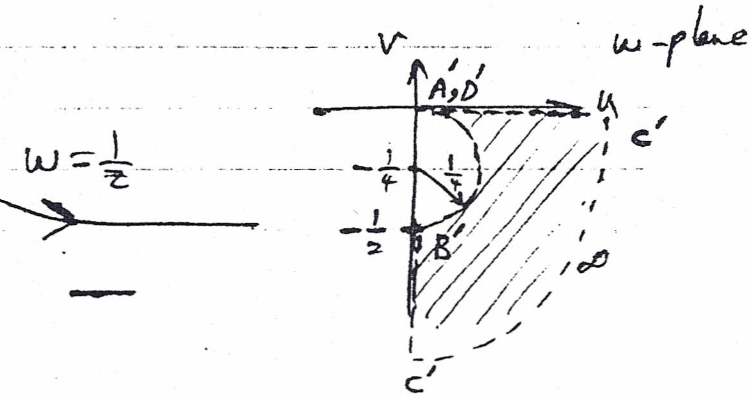
$\therefore$ The line $y = 1$ maps to $u = x + 1 \;\&\; v = 1 - x \Rightarrow x = u - 1 = 1 - v$

$\therefore\; u - 1 = 1 - v \Rightarrow u + v = 2$, This is the boundary line, the region $y > 1$ will go to $u + v > 2$ as indicated above.

$\frac{5}{17}$



=4 $w = \frac{1}{z}$



\# $w = z^2 = x^2 - y^2 + 2xyi$

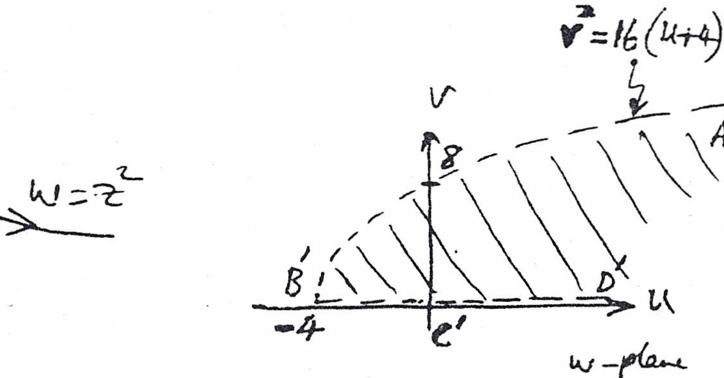
$\therefore\ u = x^2 - y^2$

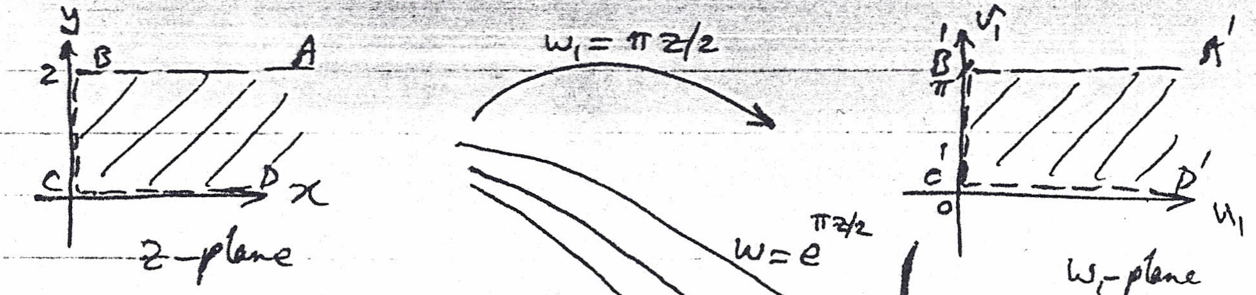
& $v = 2xy$

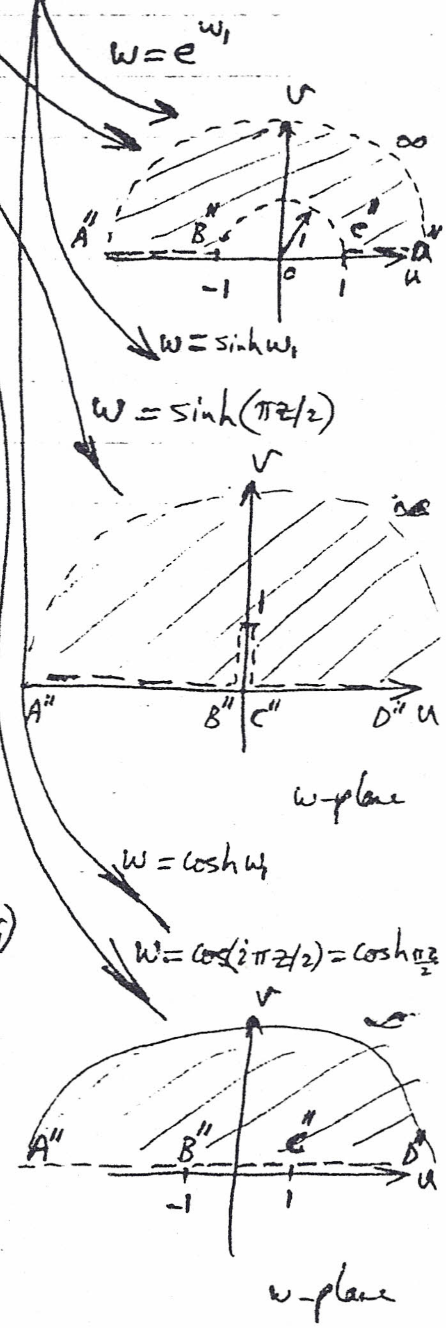
$\therefore\ y = 0,\ x > 0 \Rightarrow v = 0\ \ u > 0$

& $x = 0,\ y > 0 \Rightarrow v = 0\ \ u < 0$

& $y = 2,\ x > 0 \Rightarrow v = 4x > 0$ & $u = \left(\frac{v}{4}\right)^2 - 4$, parabola (upper half).







\# $w = e^{\pi z/2} = e^{w_1} = e^{u_1}\angle v_1$

$\therefore\ u = e^{u_1}\cos v_1$

& $v = e^{u_1}\sin v_1 > 0$

$\therefore\ v_1 = 0,\ u_1 > 0 \Rightarrow v = 0,\ u = e^{u_1} \in (1,\infty)$

& $u_1 = 0,\ v_1 \in (0,\pi) \Rightarrow |w| = 1,\ v > 0\ \therefore \angle w \in (0,\pi)$

& $v_1 = \pi,\ u_1 > 0 \Rightarrow v = 0,\ u = -e^{u_1} \in (-1,-\infty)$

OR

\# $w = \sinh\left(\frac{\pi z}{2}\right) = \sinh(w_1) = \sinh(u_1 + iv_1)$

$\therefore\ u = \sinh u_1 \cos v_1$

& $v = \cosh u_1 \sin v_1 > 0$

$\therefore\ v_1 = 0,\ u_1 > 0 \Rightarrow v = 0,\ u = \sinh u_1 \in (0,\infty)$

& $u_1 = 0,\ v_1 \in (0,\pi) \Rightarrow u = 0,\ v = \sin v_1 > 0\ \&\ \in (0,1)$

& $v_1 = \pi,\ u_1 > 0 \Rightarrow v = 0,\ u = -\sinh u_1 \in (-\infty,0)$

OR

\# $w = \cos i\pi z/2 = \cosh \pi z/2 = \cosh w_1 = \cosh(u_1 + iv_1)$

$\therefore\ u = \cosh u_1 \cos v_1$

& $v = \sinh u_1 \sin v_1 > 0$

$\therefore\ v_1 = 0,\ u_1 > 0 \Rightarrow v = 0,\ u = \cosh u_1 \in (1,\infty)$

& $u_1 = 0,\ v_1 \in (0,\pi) \Rightarrow v = 0,\ u = \cos v_1 \in (-1,1)$

& $v_1 = \pi,\ u_1 > 0 \Rightarrow v = 0,\ u = -\cosh u_1 \in (-\infty,-1)$

$\frac{9}{77}$ ④ Lets find the image of (x,y) first.

$\therefore \quad w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = u+iv$

$\therefore \quad (x,y)$ maps onto $\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = (u,v)$

$\therefore$ The line $y=a$ maps onto:

$(u,v) = \left(\frac{x}{x^2+a^2}, \frac{-a}{x^2+a^2}\right) \Longrightarrow u^2+v^2 = \frac{x^2+a^2}{(x^2+a^2)^2} = \frac{1}{x^2+a^2} = \frac{v}{-a}$

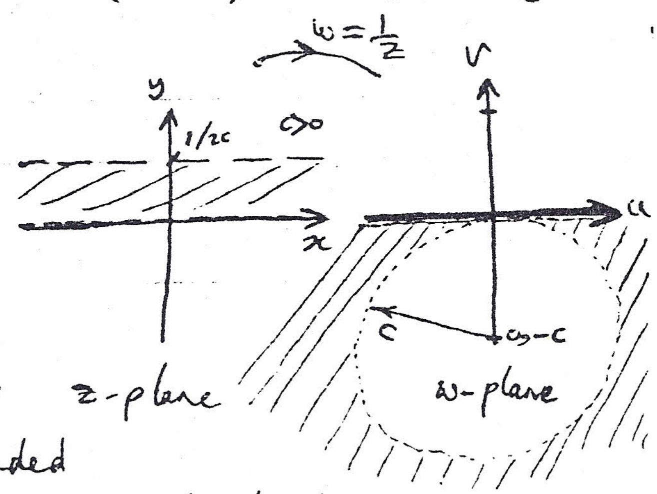
$\therefore \quad y=a$ maps onto $u^2+v^2+\frac{v}{a}=0$ or $u^2+\left(v+\frac{1}{2a}\right)^2 = \left(\frac{1}{2a}\right)^2$

$\therefore \quad y=a$ maps onto the circle centre $(0,-\frac{1}{2a})$ radius $\frac{1}{2a}$ passing through O.

$\therefore \quad y=0$ maps onto $v=0$

& $y=\frac{1}{2c}$ maps onto circle at $(0,-c)$ as centre and having c as radius & which passes through O.

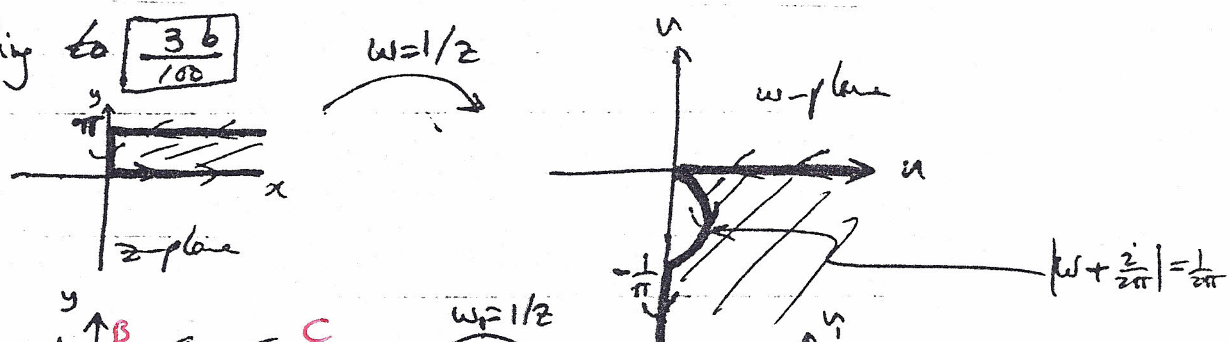
$\therefore \quad y \in (0, \frac{1}{2c})$ maps onto the region between $v=0$ & the above mentioned circle as shown in here.



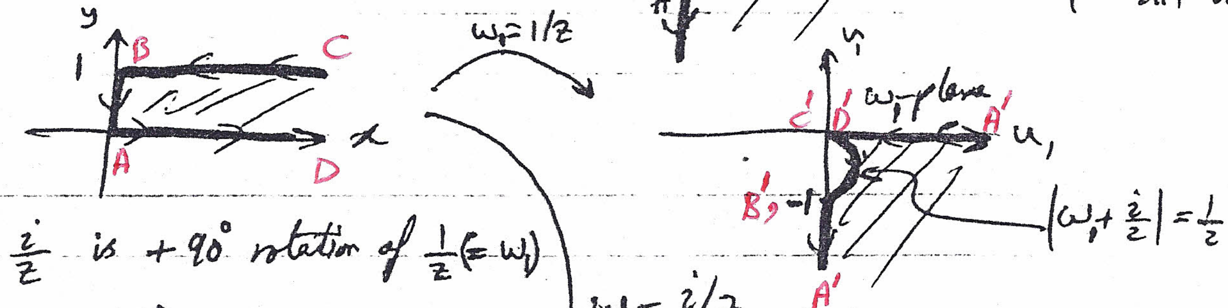
The circle & line $v=0$ are not included in the region because $y=0$ & $y=\frac{1}{2c}$ are not included.

$\frac{14}{78}$ ② According to $\boxed{\frac{36}{100}}$

$\therefore$

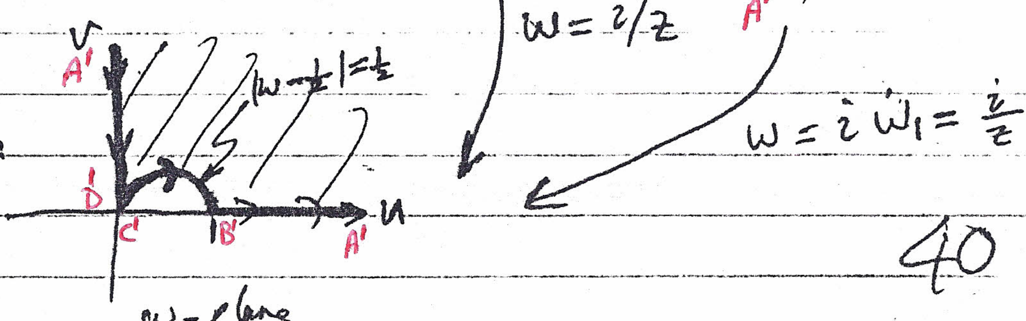


$\therefore$



$\because \quad w = \frac{i}{z}$ is $+90°$ rotation of $\frac{1}{z} (= w_1)$

$\therefore$ The map is:



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83 (4) $w = \dfrac{az+b}{cz+d}$

$\therefore\ czw + dw - az - b = 0$

$\because$ 2 maps onto 1 $\quad \therefore\ c*2*1 + d*1 - a*2 - b = 0$

$\therefore\ 2c + d - 2a - b = 0 \quad (1)$

$\because$ i maps onto i $\quad \therefore\ c*i*i + d*i - a*i - b = 0$

$\therefore\ -c + id - ia - b = 0 \quad (2)$

$\because$ -2 maps onto -1 $\quad \therefore\ c*(-2)*(-1) + d*(-1) - a*(-2) - b = 0$

$\therefore\ 2c - d + 2a - b = 0 \quad (3)$

Putting (1), (2) & (3) in matrix form:

$$\therefore \begin{bmatrix} 2 & 1 & -2 & -1 \\ -1 & i & -i & -1 \\ 2 & -1 & +2 & -1 \end{bmatrix} \begin{bmatrix} c \\ d \\ a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ Solving the system:}$$

$$\therefore \begin{bmatrix} 2 & 1 & -2 & -1 \\ -1 & i & -i & -1 \\ 2 & -1 & 2 & -1 \end{bmatrix} \xrightarrow[R_1 - R_3]{-R_2,\ R_1 + 2R_2} \begin{bmatrix} 1 & -i & i & 1 \\ 0 & 1+2i & -2-2i & -3 \\ 0 & 2 & -4 & 0 \end{bmatrix} \xrightarrow[2R_2 - (1+2i)R_3]{R_3/2} \begin{bmatrix} 1 & -i & i & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4i & -6 \end{bmatrix}.$$

$\therefore\ 4ia - 6b = 0 \quad \therefore\ a = \dfrac{6b}{4i} = -3bi/2$

& $d - 2a = 0 \quad \therefore\ d = 2a = -3bi$

& $c - id + ia + b = 0 \quad \therefore\ c = id - ia - b = -3bi^2 + 3bi^2/2 - b = 3b - \frac{3b}{2} - b$
$= b/2$

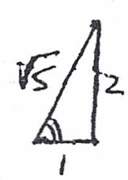
$$\therefore\ w = \frac{az+b}{cz+d} = \frac{-\frac{3bi}{2}.z + b}{\frac{b}{2}.z - 3bi} = \frac{3iz - 2}{-z + 6i} = \frac{3z + 2i}{iz + 6}$$

(check: $w(2) = \dfrac{3*2 + 2i}{i*2 + 6} = \dfrac{6 + 2i}{6 + 2i} = 1$, ok

$w(i) = \dfrac{3i + 2i}{-1 + 6} = \dfrac{5i}{5} = i$, ok

$w(-2) = \dfrac{3*(-2) + 2i}{i*(-2) + 6} = \dfrac{-6 + 2i}{6 - 2i} = -1$, ok).

$$\# \quad \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$



$\because\ z_1 = 2,\ z_2 = i,\ z_3 = -2,\ w_1 = 1, w_2 = i, w_3 = -1$

$$\therefore\ \frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-2)(i+2)}{(z+2)(i-2)}$$

$$\therefore\ \left(\frac{w-1}{w+1}\right)\frac{\sqrt{2}\angle \pi/4}{\sqrt{2}\angle \pi - \pi/4} = \left(\frac{z-2}{z+2}\right)\frac{\sqrt{5}\angle \tan^{-1}\frac{1}{2}}{\sqrt{5}\angle \pi - \tan^{-1}\frac{1}{2}} \Leftrightarrow \left(\frac{w-1}{w+1}\right) = \left(\frac{z-2}{z+2}\right)\angle^{-\pi + \tan^{-1}\frac{1}{2}}\ \tan^{-1}\tfrac{1}{2} - \tfrac{\pi}{4} + \pi - \tfrac{\pi}{4}$$

$$= \left(\frac{z-2}{z+2}\right)\angle -\frac{\pi}{2} + \tan^{-1}\frac{1}{2} = \left(\frac{z-2}{z+2}\right)\angle -\frac{\pi}{2} + 2(\pi/2 - \tan^{-1} 2) = \left(\frac{z-2}{z+2}\right)\angle \frac{\pi}{2} - 2\tan^{-1} 2$$

$$= \left(\frac{z-2}{z+2}\right)\left(\cos\left(\frac{\pi}{2} - 2\tan^{-1} 2\right) + i\sin\left(\frac{\pi}{2} - 2\tan^{-1} 2\right)\right) = \left(\frac{z-2}{z+2}\right)\left(\sin(2\tan^{-1} 2) + i\cos(2\tan^{-1} 2)\right)$$

$$= \left(\frac{2-z}{z+2}\right) * \left(2\sin\tan^{-1}2\cos\tan^{-1}2 + i\left(\cos^2\tan^{-1}2 - \sin^2\tan^{-1}2\right)\right) = \left(\frac{2-z}{2+z}\right)\left[2\,\frac{2}{\sqrt5}\cdot\frac{1}{\sqrt5} + i\left(\frac{1}{5}-\frac{4}{5}\right)\right] =$$

$$= \left(\frac{2-z}{2+z}\right)\left(\frac{4}{5} - \frac{3i}{5}\right)(-1)$$

$$\therefore 5(w-1)(2+z) = (2-z)(w+1)(4-3i)(-1)$$

$$\therefore 5wz + 10w - 10 - 5z = (2w + 2 - wz - z)(4-3i)(-1)$$

$$\therefore w\left(5z + 10 + (2-z)(4-3i)\right) = (z-2)(4-3i) + 10 + 5z$$

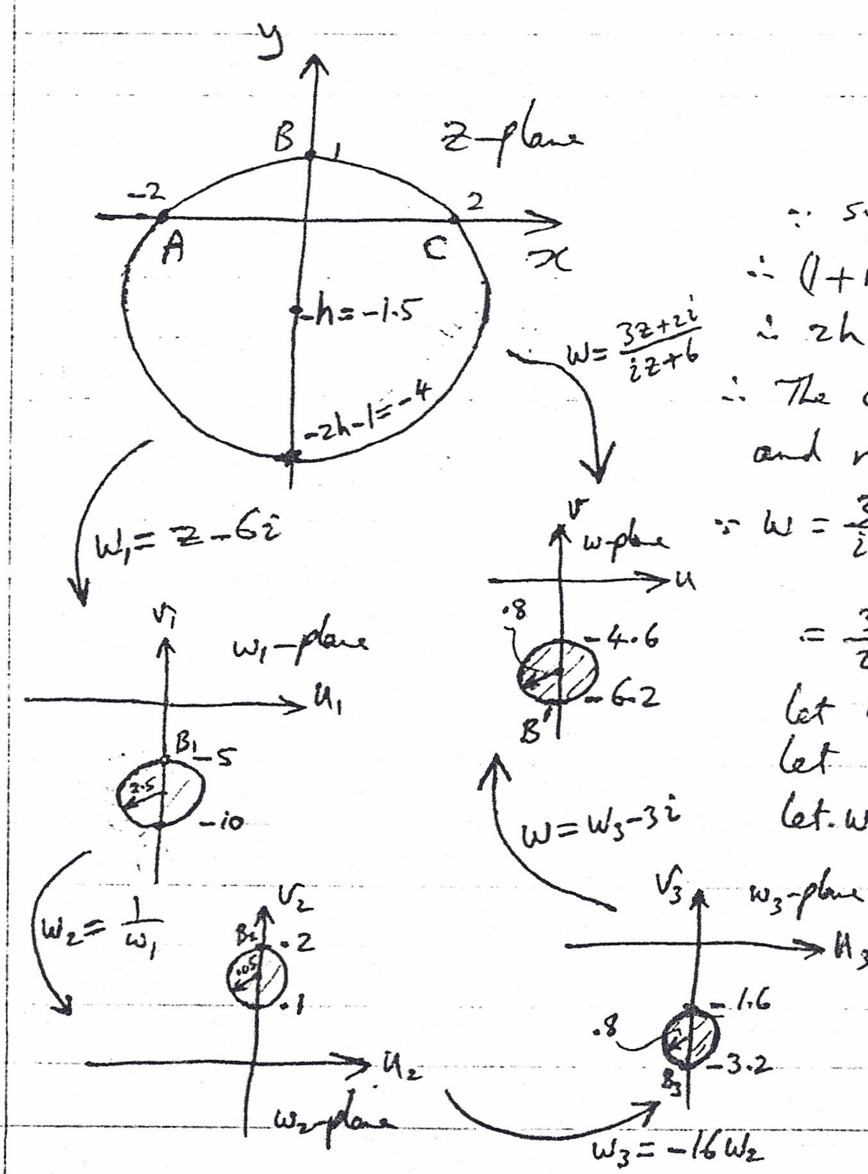
$$\therefore w(5z + 10 + 8 - 6i - 4z + 3iz) = (4 - 3i + 5)z - 8 + 6i + 10$$

$$\therefore w\left((1+3i)z + 18 - 6i\right) = (9-3i)z + 2 + 6i = (1+3i)(-3iz+2)$$

$$\therefore w = \frac{(1+3i)(2-3iz)}{6(3-i) + z(1+3i)} = \frac{(1+3i)(2-3iz)}{-6i(1+3i) + z(1+3i)} = \frac{2-3iz}{z-6i}.$$

$\therefore$ The transformation is $w = \dfrac{2-3iz}{z-6i} = \dfrac{3z+2i}{iz+6}$

\#



$\therefore$ symmetric about y $\therefore$ Centre at y

$\therefore (1+h)^2 = 2^2 + h^2 \quad \therefore 1 + 2h + h^2 = 4 + h^2$

$\therefore 2h = 3 \quad \therefore h = 1.5$

$\therefore$ The circle has centre at $(0,-1.5)$ and radius of 2.5

$$\therefore w = \frac{3z+2i}{iz+6} = \frac{\frac{3}{i}(iz+6) - 16i}{iz+6}$$

$$= \frac{3}{i} - \frac{16i}{iz+6} = -3i - \frac{16}{z-6i}$$

Let $w_1 = z - 6i \quad \therefore w = -3i - 16/w_1$

Let $w_2 = \frac{1}{w_1} \quad \therefore w = -3i - 16w_2$

Let $w_3 = -16w_2 \quad \therefore w = -3i + w_3$

$\therefore$ Image is a circle radius $.8$ centre $(0,-5.4)$.

2/83

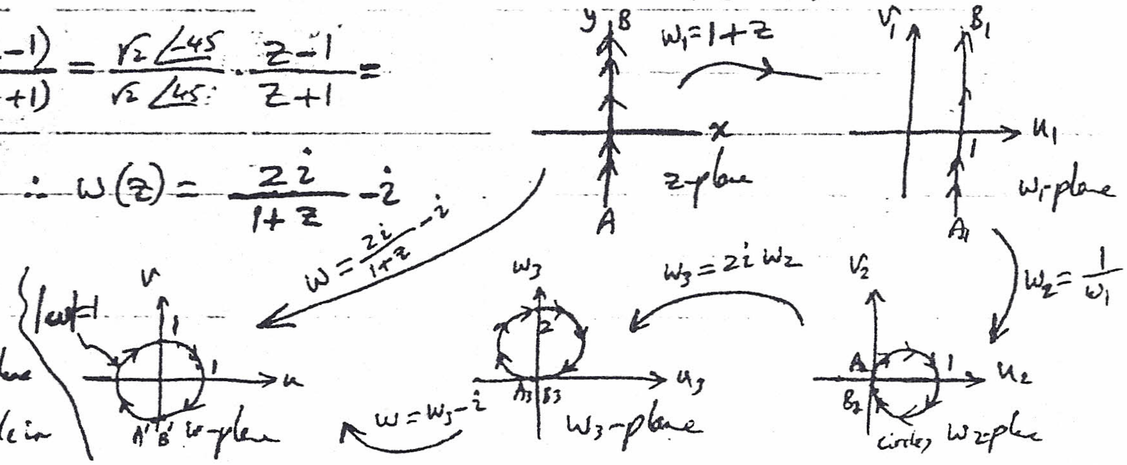
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \Rightarrow \frac{(w+1)(i-1)}{(w-1)(i+1)}=\frac{(z+i)(0-i)}{(z-i)(0+i)}$$

$$\Rightarrow \left(\frac{w+1}{w-1}\right)\frac{\sqrt{2}\angle 135^\circ}{\sqrt{2}\angle 45^\circ}=-\left(\frac{z+i}{z-i}\right) \Rightarrow i(w+1)(z-i)=-(w-1)(z+i) \Rightarrow$$

$$\therefore w[z(i+1)+1+i]=-iz+z+i-1 \Rightarrow w=\frac{z(1-i)-(1-i)}{(1+i)(z+1)}=$$

$$=\frac{(1-i)(z-1)}{(1+i)(z+1)}=\frac{\sqrt{2}\angle -45}{\sqrt{2}\angle 45}\cdot\frac{z-1}{z+1}=$$

$$=-i\,\frac{z-1}{z+1} \quad \therefore w(z)=\frac{2i}{1+z}-i$$



$\therefore$ The image of y axis in z-plane is the unit circle in w-plane. $\{|w|=1$

3/83 (3)

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad \because z_1=\infty \Rightarrow w_1=0$$

$\therefore \frac{1}{z_1}=0$ & $w_1=0$ & $\because w(0)=\infty$ $\therefore z_3=0 \Rightarrow w_3=\infty \Rightarrow \frac{1}{w_3}=0$

& $z_2=w_2=i$

$\therefore$ The transformation can be rewritten as $\dfrac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\dfrac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$

Substituting the points

$$\therefore \frac{(w-0)(i*0-1)}{(w*0-1)(i-0)}=\frac{(z*0-1)(i-0)}{(z-0)(i*0-1)} \Rightarrow \frac{-w}{-i}=\frac{-i}{-z} \quad \therefore w=\frac{-1}{z}$$

7/83 (a) $z=w=\frac{z-1}{z+1}$ $\quad\therefore z^2+\cancel{z}=\cancel{z}-1 \quad \therefore z^2=-1 \quad \therefore z=\pm i$

$\therefore$ The fixed points of w are $\pm i$.

(b) $z=w=\frac{6z-9}{z}$ $\quad\therefore z^2=6z-9 \quad \therefore z^2-6z+9=0 \quad \therefore (z-3)^2=0 \quad \therefore z=3$

$\therefore$ The fixed points of w are only 3.

$\frac{11}{84}$ $\quad w = e^{i\alpha}\cdot\frac{z-z_0}{z-\bar{z}_0}$

∴ $w(\infty)=e^{i\alpha}=-1 \Rightarrow \therefore \alpha=(1+2k)\pi$

& $w(0)=e^{i\alpha}\cdot\frac{z_0}{\bar{z}_0}=-1\cdot\frac{z_0}{\bar{z}_0}=1 \Rightarrow \therefore z_0=-\bar{z}_0 \quad \therefore \operatorname{Re} z_0=0$

& $w(1)=e^{i\alpha}\cdot\frac{1-z_0}{1-\bar{z}_0}=-1\cdot\frac{1-i\,Im\,z_0}{1+i\,Im\,z_0}=i$, ∴ $i-Im\,z_0=-1+i\,Im\,z_0$

∴ $Im\,z_0=\frac{i+1}{i+1}=1$

∴ $W=-1\cdot\frac{z-i}{z+i}=\frac{i-z}{i+z}$

(check: $w(\infty)=-1$, ∴ OK, $w(0)=1$ ∴ OK, $w(1)=\frac{i-1}{i+1}=\frac{\sqrt{2}\angle 135^\circ}{\sqrt{2}\angle 45^\circ}=\angle 90^\circ=$ ∴ OK)

∴ The above transformation maps the x-axis into $|w|=1$

(check: $|w(y=0)|=\left|\frac{i-x}{i+x}\right|=\frac{|i-x|}{|i+x|}=\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}=1$, ∴ OK)

& The above transformation maps the region $y>0$ into $|w|<1$

(check: $|w(y=a^2)|=\left|\frac{i-(x+ia^2)}{i+(x+ia^2)}\right|=\frac{|-x+i(1-a^2)|}{|+x+i(1+a^2)|}=\frac{\sqrt{x^2+(1-a^2)^2}}{\sqrt{x^2+(1+a^2)^2}}<1$ for $a\neq 0$, ∴ OK)

The immage of the positive x-axis ($y=0$, $x=a^2$, $a\neq 0$) will then be

$w(a^2+0i)=\frac{i-(a^2+0i)}{i+(a^2+0i)}=\frac{i-a^2}{i+a^2}=\frac{(i-a^2)}{(i+a^2)}\cdot\frac{(i-a^2)}{(i-a^2)}=\frac{(i-a^2)^2}{i^2-a^4}=-\frac{(-1+a^4-2a^2 i)}{1+a^4}=u+iv$

∴ $u=\frac{-(-1+a^4)}{1+a^4}=\frac{1-a^4}{1+a^4}=1-\frac{2a^4}{1+a^4}$ $\left(\begin{array}{l}\text{Can be +ve at } a^2(=x)\in(0,1)\\ \quad " \quad " \quad 0 \text{ at } a^2(=x)=1\\ \quad " \quad " \quad \text{-ve at } a^2(=x)\in(1,\infty)\end{array}\right)$

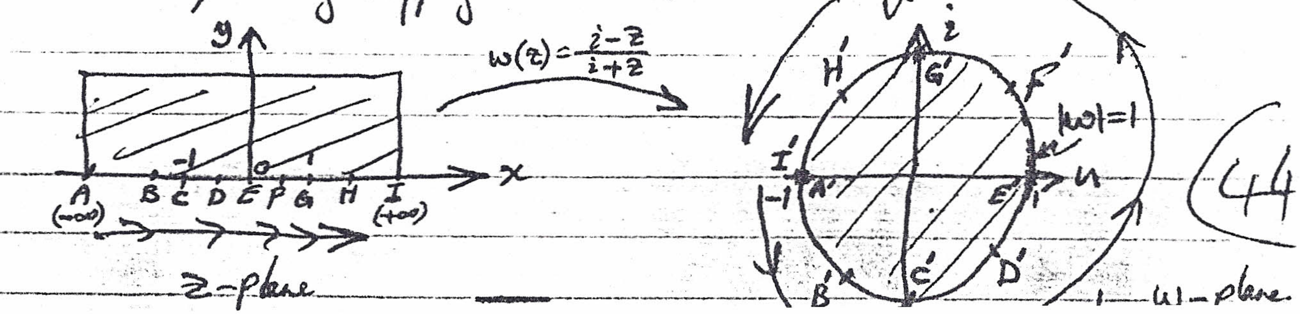
∴ $v=\frac{2a^2}{1+a^4}$ always >0

∴ The immage of the positive x-axis goes to the positive-v-half of $|w|=1$.

Similarly, the immage of the negative x-axis ($y=0$, $x=-a^2$, $a\neq 0$) is given by $u+iv=\frac{1-a^4}{1+a^4}+i\left(\frac{-2a^2}{1+a^4}\right)$

Again u $\left\langle\begin{array}{l}\text{+ve at } -a^2(=x)\in(-1,0)\\ 0 \quad " \quad -a^2(=x)=-1\\ \text{-ve} \quad " \quad -a^2(=x)\in(-\infty,-1)\end{array}\right.$ & v is always -ve

Thus the following mapping is obtained.



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$$W = \frac{iz+\exp(i\pi/4)}{z+\exp(i\pi/4)} = \frac{[\exp(i\pi/2)]\, z - [\exp(i\pi/2)]\cdot[\exp(i\pi)]\cdot[\exp(-i\pi/4)]}{z - \exp(i\pi)\cdot\exp(i\pi/4)} = e^{i\frac{\pi}{2}} \cdot \frac{z - e^{i3\pi/4}}{z - \bar{e}^{i3\pi/4}}$$

$\therefore$ $w(z)$ is in the form $\exp(i\alpha)\cdot\frac{z-z_0}{z-\bar{z}_0}$ where $z_0 = e^{i3\pi/4}$, $\alpha = \frac{\pi}{2}$

f $\because$ $\mathrm{Im}\, z_0 = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}} > 0$

$\therefore$ $w(z)$ maps the disk $y = \mathrm{Im}\, z > 0$ onto the interior of $|w| = 1$

(check: $z = e^{i\pi/4} \in \{\mathrm{Im}\, z > 0\}$ and maps onto $\frac{i\cdot e^{i\pi/4} + e^{i\pi/4}}{2e^{i\pi/4}} = \frac{1+i}{2}$ whose magnitude $= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} < 1$, $\therefore$ OK)

Now, consider the line $y=0$ (x-axis):

$\therefore$ $w(z) = w(x+i0) = \frac{ix + \angle\pi/4}{x + \angle\pi/4} = \frac{ix + \frac{1}{\sqrt{2}}(1+i)}{x + \frac{1}{\sqrt{2}}(1+i)} = \frac{1 + i(1+x\sqrt{2})}{(1+x\sqrt{2}) + i}$

w is supposed to map $y=0$ into $|w|=1$ because of its nature.

(This is true, because $|w|^2 = \frac{1+(1+x\sqrt{2})^2}{(1+x\sqrt{2})^2+1} = 1$ $\therefore |w| = 1$)

$\therefore$ $w(z \text{ on the } x\text{-axis}) = \frac{\sqrt{1+(1+x\sqrt{2})^2}\ \angle\tan^{-1}(1+x\sqrt{2})}{\sqrt{(1+x\sqrt{2})^2+1}\ \angle\cot^{-1}(1+x\sqrt{2})} = \angle\tan^{-1}(1+x\sqrt{2}) - \cot^{-1}(1+x\sqrt{2})$

but $\tan^{-1} r + \cot^{-1} r = \pi/2$

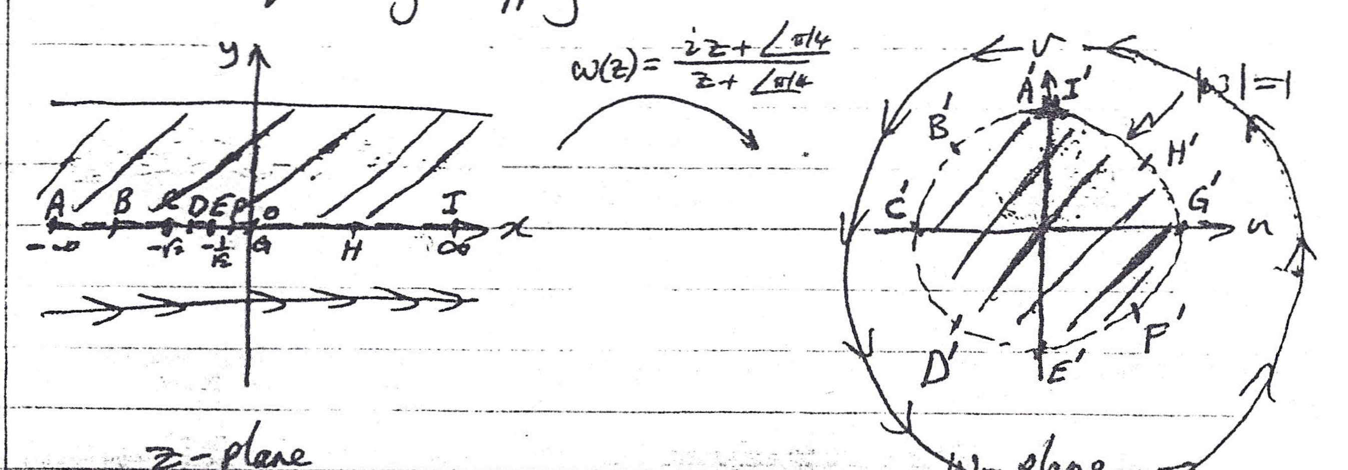
$\therefore$ $\cot^{-1} r = \pi/2 - \tan^{-1} r$

$\therefore$ $w(x\text{-axis}) = \angle 2\tan^{-1}(1+x\sqrt{2}) - \pi/2 = w(x)$

Hence, we can see the following map from $y=0$ onto $|w|=1$:

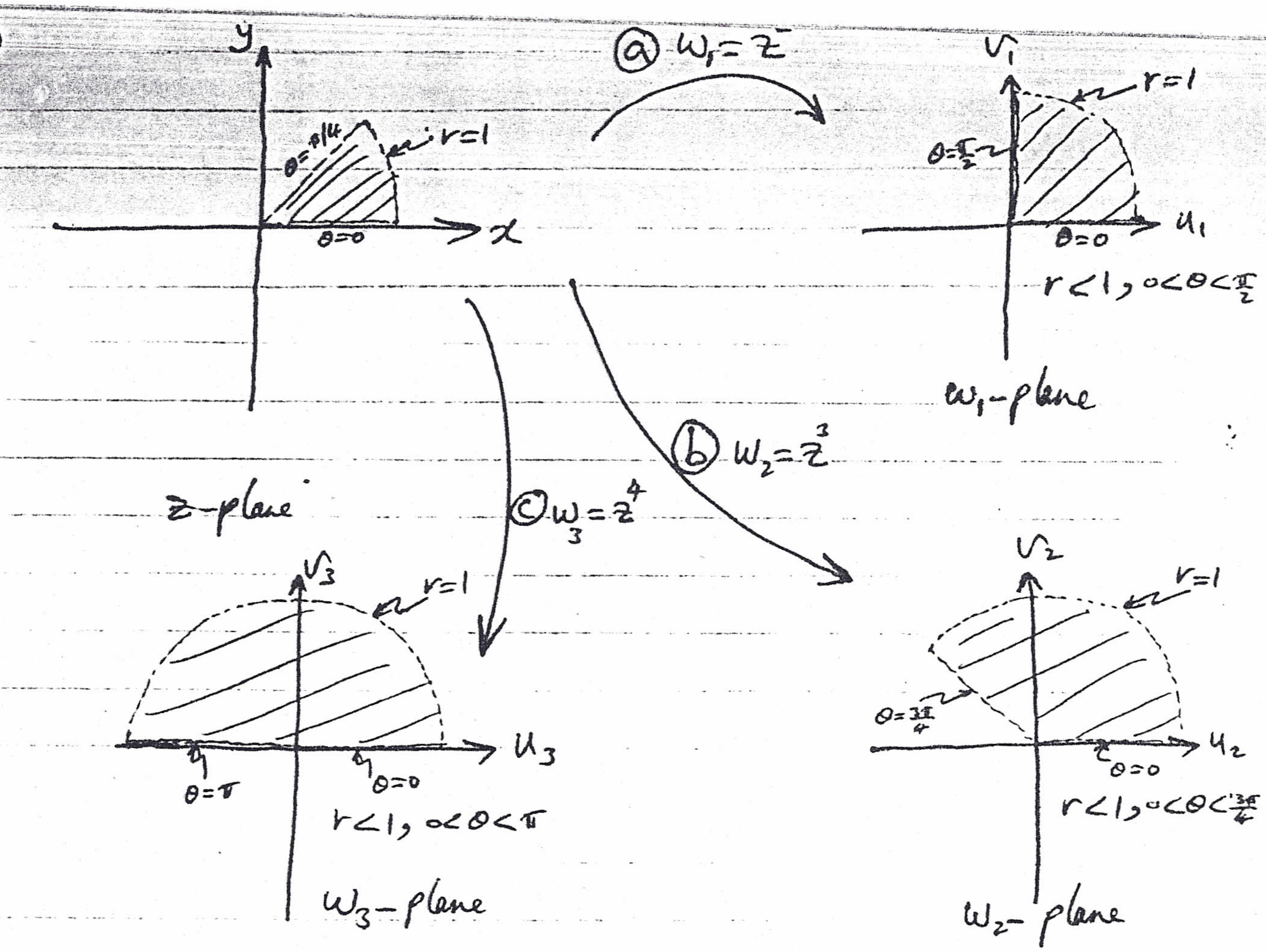
x	$-\infty$	$-\sqrt{2}$	$-1/\sqrt{2}$	0	$+\infty$
$\angle w(x)$	$-3\pi/2$	$-2\pi/2$	$-\pi/2$	0	$\pi/2$

Thus, the following mapping is obtained:



(Note: you can prove that $y = \frac{1}{\sqrt{2}}$ maps onto $|w - \frac{i}{2}| = \frac{1}{2}$)

$\frac{1}{93}$ (12)



$\frac{2}{93}$ $\quad w(x+iy) = w(z) = z^2 = (x+iy)^2 = x^2-y^2+2xyi = u+iv$

$\therefore$ $u = x^2-y^2$ & $v = 2xy$

ⓐ Lines $x=c$, will map to:

$u = c^2-y^2$ & $v = 2cy \Rightarrow u = c^2-\left(\frac{v}{2c}\right)^2 \Rightarrow v^2 = -4c^2(u-c^2)$

$\therefore$ vertix at $(c^2,0)$, axis along negative u, focal length $p = \left|\frac{-4c^2}{4}\right| = c^2$

$\therefore$ Focus = Vertix + P = $\langle c^2,0\rangle + c^2\langle -1,0\rangle = \langle 0,0\rangle$, i.e. O.

ⓑ Lines $y=d$, will map to:

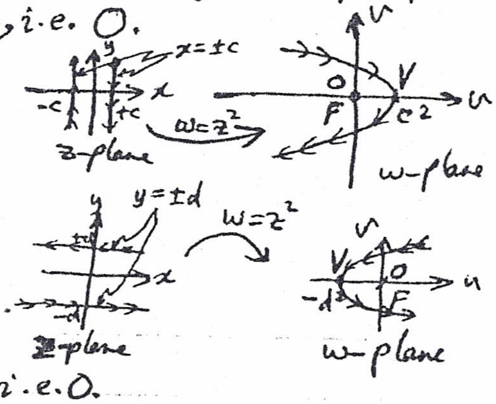
$u = x^2-d^2$ & $v = 2xd$

$\therefore$ $u = \left(\frac{v}{2d}\right)^2 - d^2 \Rightarrow v^2 = 4d^2(u+d^2)$

$\therefore$ vertix at $(-d^2,0)$, axis along positive u,

focal length $p = \left|\frac{4d^2}{4}\right| = d^2$

$\therefore$ Focus = Vertix + P = $\langle -d^2,0\rangle + d^2\langle 1,0\rangle = \langle 0,0\rangle$, i.e. O.



$\frac{70}{94}$

$F(z)=(z^2-1)^{\frac12} = [(z-1)(z+1)]^{\frac12} = [(z-1)^{1/2}\cdot(z+1)^{1/2}]$

$= \exp\{\log[(z-1)^{1/2}(z+1)^{\frac12}]\} = \exp\{\log[(r_1\angle\theta_1)^{1/2}\cdot(r_2\angle\theta_2)^{1/2}]\}$

$= \exp\{\log(r_1\angle\theta_1)^{\frac12} + \log(r_2\angle\theta_2)^{\frac12}\}$

$= \exp\{\log(\sqrt{r_1}\angle\tfrac{\theta_1}{2}) + \log(\sqrt{r_2}\angle\theta_2/2)\}$

$= [\exp(\log(\sqrt{r_1}\angle\theta_1/2))]\cdot[\exp(\log(\sqrt{r_2}\angle\theta_2/2))] =$

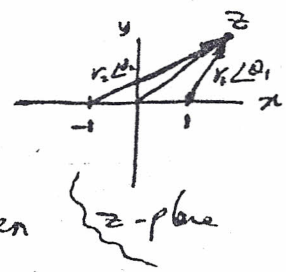
$= [\exp(\log\sqrt{r_1} + i\tfrac{\theta_1}{2})]\cdot[\exp(\log\sqrt{r_2} + i\,\theta_2/2)]$

$= \sqrt{r_1}\cdot\angle\theta_1/2\ .\ \sqrt{r_2}\cdot\angle\theta_2/2 = \sqrt{r_1 r_2}\cdot\angle(\theta_1+\theta_2)/2$

where $(z-1)=r_1\angle\theta_1$, $(z+1)=r_2\angle\theta_2$ with $r_1>0$, $r_2>0$ & $r_1+r_2>2$, $\theta_1,\theta_2\in[0,2\pi)$

provided θ_1 & θ_2 don't discontinue (and they don't in the domain given).

$\therefore$ $F(z)$ is analytic everywhere in the domain of difinition.



The branch cut is at $z:\ r_1+r_2 \le 2$

From the diagram r_1+r_2 is always >2 except when z is on the segment of the x-axis between -1 & $+1$ where $r_1+r_2=2$ always

$\therefore$ The branch cut of $F(z)$ is at $y=0$ & $x\in[-1,1]$

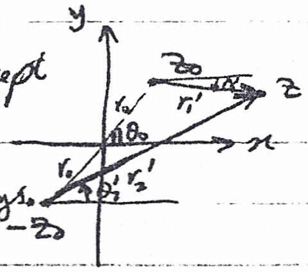
(Note: $F(z)$ (when $z=x+i0^+$, $x\in[-1,1]$) $=\sqrt{r_1r_2}\ \angle(\pi+0)/2 = i\sqrt{r_1r_2}$ & (when $z=x+i0^-$) $=\sqrt{r_1r_2}\ \angle(\pi+2\pi)/2 = -i\sqrt{r_1r_2}$)

Now, $F_0(z)=(z^2-z_0^2)^{\frac12} = [(z-z_0)(z+z_0)]$

$=$ (in similar argument as above) $\sqrt{r_1'r_2'}\cdot\angle(\theta_1'+\theta_2')/2$

where $z_0=r_0\angle\theta_0$, $z-z_0=r_1'\angle\theta_1'$, $z+z_0=r_2'\angle\theta_2'$ with $r_1'>0$, $r_2'>0$ & $r_1'+r_2'>2r_0$, $\theta_1',\theta_2'\in[\theta_0, 2\pi+\theta_0)$

$\therefore$ The branch cut is at $z:\ r_1'+r_2'\le 2r_0$



From diagram $r_1'+r_2'$ is always $>2r_0$ except when z is on the segment connecting z_0 & $-z_0$ where $r_1'+r_2'=2r_0$ always.

$\therefore$ The branch cut of $F_0(z)$ is at

$y=\frac{y_0}{x_0}x$ & $x\in[-x_0,x_0]$ (where $x_0+iy_0=z_0$)

(Note: when z is just above this segment then

$F_0(z)=\sqrt{r_1'r_2'}\cdot\angle[(\pi+\theta_0)+(\theta_0)]/2 = \sqrt{r_1'r_2'}\ \angle\theta_0+\pi/2 = i\sqrt{r_1'r_2'}\ \angle\theta_0$

& when z is just below this segment then $F_0(z)=\sqrt{r_1'r_2'}\ \angle[(\pi+\theta_0)+(2\pi+\theta_0)]/2 =$

$=\sqrt{r_1'r_2'}\ \angle\tfrac{3\pi}{2}+\theta_0 = -i\sqrt{r_1'r_2'}\cdot\angle\theta_0$; This is why it is not analytic here).

Letting $Z_1=\frac{z}{z_0}$ $\therefore$ $F(Z_1)=[(\tfrac{z}{z_0})^2-1]^{1/2} = (\tfrac{z^2-z_0^2}{z_0^2})^{1/2} = \frac{(z^2-z_0^2)^{1/2}}{z_0} = \frac{F_0(z)}{z_0}$

$\therefore$ $F_0(z)=z_0\cdot F(Z_1)$, $z_0\neq0$

$\boxed{\#}$ $f(z) = z^2 = (x+iy)^2$

$= x^2 - y^2 + 2xyi$

$\therefore$ $u(x,y) = x^2 - y^2$

& $v(x,y) = 2xy$

$\therefore$ Segment along x: $y=0$, $x \in [0, A]$

$\therefore$ Immage is $v=0$ & $x \in [0, A^2]$

& Segment along y: $x=0$, $y \in [0, A]$

$\therefore$ Immage is $v=0$ & $x \in [-A^2, 0]$

& Segment along $x+y=A$: $x \in [0, A]$ & $y \in [0, A]$

$\therefore$ $y = A - x$ & it goes to:

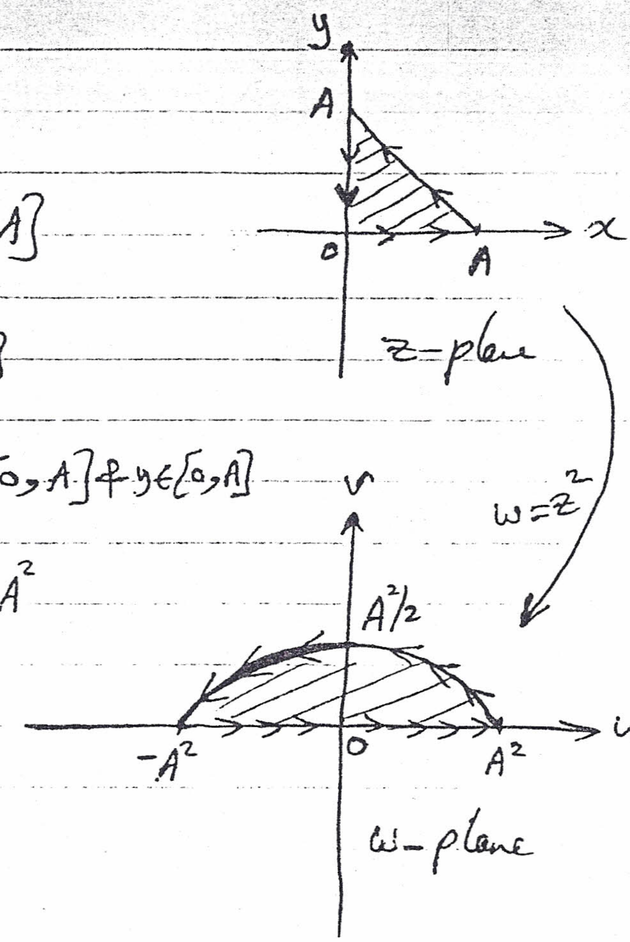
$\therefore$ $u = x^2 - y^2 = x^2 - (A-x)^2 = 2Ax - A^2$

$\therefore$ $x = (A^2 + u)/2A$

& $v = 2xy = 2x(A-x)$

$$= 2 \cdot \left(\frac{A^2+u}{2A}\right)\left(A - \frac{A^2+u}{2A}\right)$$

$$= \frac{A^2+u}{A} \cdot \frac{A^2-u}{2A} = \frac{A^4-u^2}{2A^2}$$



$\therefore$ $u^2 = 2A^2\left(\frac{A^2}{2} - v\right)$ $\therefore$ parabola & $v \geq 0$ & axis along $-v$, vertex at $(0, A^2/2)$. Hence, immage is as shown above.

$\boxed{\frac{3}{100}}$ $w = e^z$, Consider the boundaries $x=0$, $y=0$, $y=\pi$, $x=\infty$

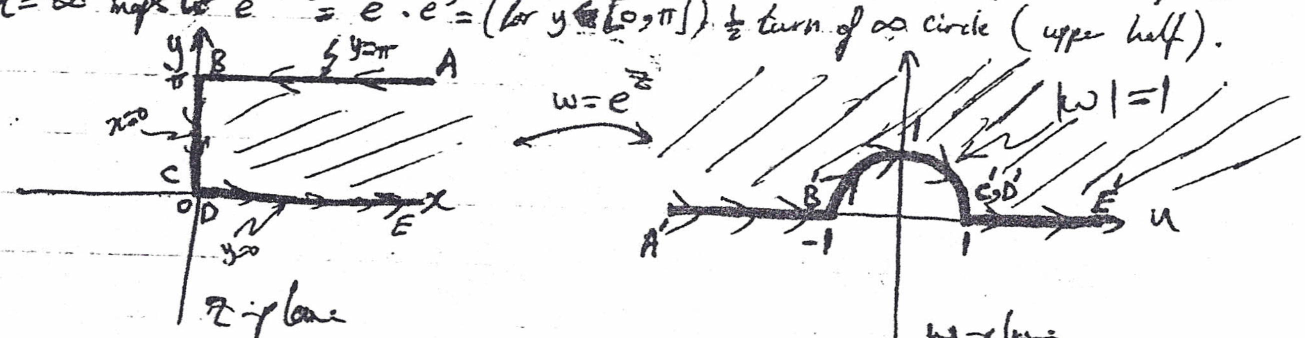
$\therefore$ $x=0$ maps to $e^{0+iy} = e^{iy} =$ (for $y \in [0, \pi]$) $\frac{1}{2}$ turn of unit circle (upper half)

& $y=0$ maps to $e^{x+i0} = e^x = u+iv$ $\therefore$ $v=0$ & u (for $x \in [0, \infty)$) $\in [1, \infty)$

& $y=\pi$ maps to $e^{x+i\pi} = e^x(-1+i0) = -e^x = u+iv$ $\therefore$ $v=0$ & u (for $x \in [0,\infty)$) $\in [-1, \infty)$

& $x=\infty$ maps to $e^{\infty+iy} = e^{\infty} \cdot e^{iy} =$ (for $y \in [0, \pi]$) $\frac{1}{2}$ turn of ∞ circle (upper half).

$\therefore$



$w=\frac{1}{z}$, consider the boundaries $x=0,\ y=0,\ y=\pi,\ x=\infty$.

$\therefore$ $x=0$ maps to $\frac{1}{0+iy}=\frac{-i}{y}=u+iv$ $\therefore u=0$ & $v=-\frac{1}{y}$

$\therefore$ v (for $y\in[0,\pi]$) $\in\left[-\frac{1}{\pi},-\infty\right)$

& $x=\infty$ maps to $\frac{1}{\infty+iy}=0=u+iv$ $\therefore$ $u=0$ & $v=0$

& $y=0$ maps to:

$$\frac{1}{x+i0}=\frac{1}{x}=u+iv \quad \therefore v=0 \ \& \ u=\frac{1}{x}$$

$\therefore$ u (for $x\in[0,\infty)$) $\in(0,\infty)$

& $y=\pi$ maps to: $\frac{1}{x+i\pi}=\frac{x-i\pi}{x^2+\pi^2}=u+iv$

$\therefore$ $u=\frac{x}{x^2+\pi^2}$ & $v=\frac{-\pi}{x^2+\pi^2}$ $\therefore$ $\frac{u}{v}=-\frac{x}{\pi}$ $\therefore x=\frac{-\pi u}{v}$

$\therefore$ $u=\frac{-\pi u/v}{(-\pi u/v)^2+\pi^2}=\frac{-\pi uv}{\pi^2u^2+\pi^2v^2}$ $\therefore$ $\pi^2(u^2+v^2)=-\pi v$

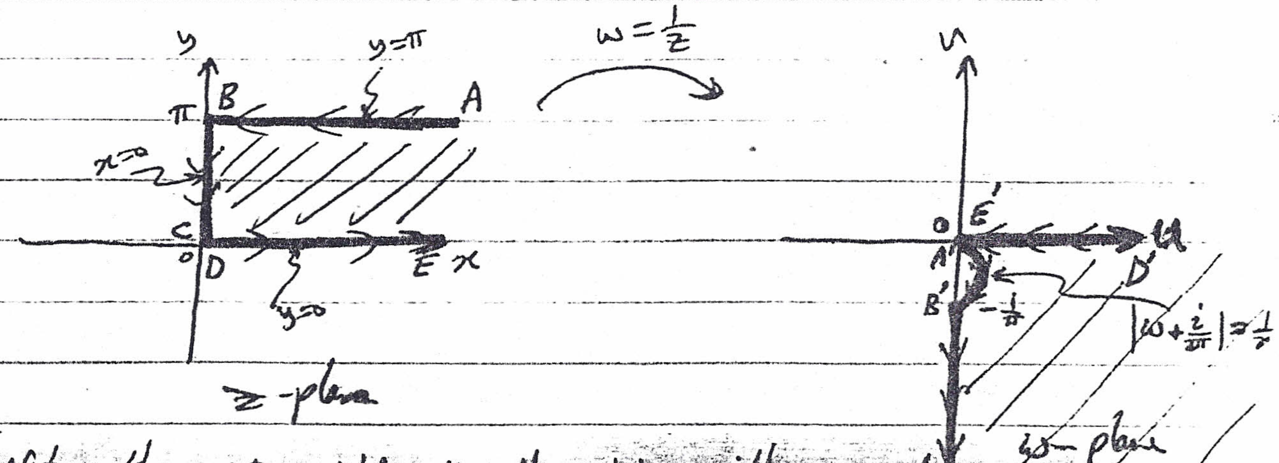
$\therefore$ $u^2+v^2+\frac{v}{\pi}=0$ or $u^2+\left(v+\frac{1}{2\pi}\right)^2=\left(\frac{1}{2\pi}\right)^2$

This is a circle centre $\left(0,\frac{-1}{2\pi}\right)$ & radius $\frac{1}{2\pi}$

$\because$ $x\in[0,\infty)$ $\therefore$ u is always positive & v is negative

$\therefore$ $y=\pi$ maps to the $+$ve half (in the fourth $\frac{1}{4}$) of circle $u^2+\left(v+\frac{1}{2\pi}\right)^2=\left(\frac{1}{2\pi}\right)^2$

$\therefore$

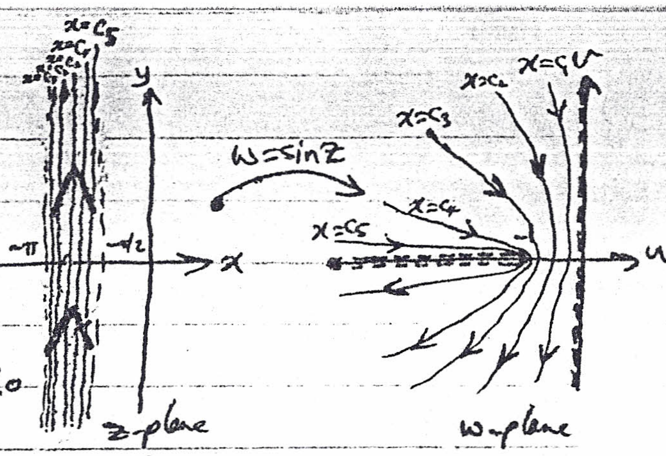


(Note: the points neighbouring the origin, will go to the ∞ circle in the fourth quarter of the w-plane)

$\frac{7}{100}$ (5) $w = \sin z = \sin(x+iy)$

$= \sin x \cosh y + i\cos x \sinh y$

$= u + iv$



$\therefore \cosh y = \frac{u}{\sin x}$, $\sinh y = \frac{v}{\cos x}$

$\because \cosh^2 y - \sinh^2 y = 1$, $x = C < 0$.

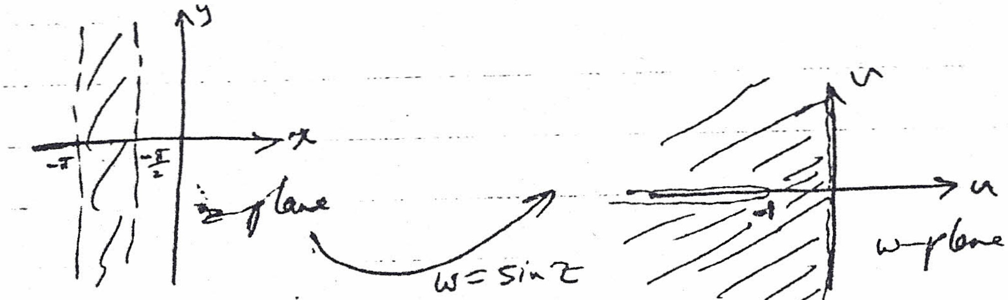
$\therefore \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$, hyperbola on $u<0$

$\because C \in (-\pi, -\pi/2)$ i.e third quadrant

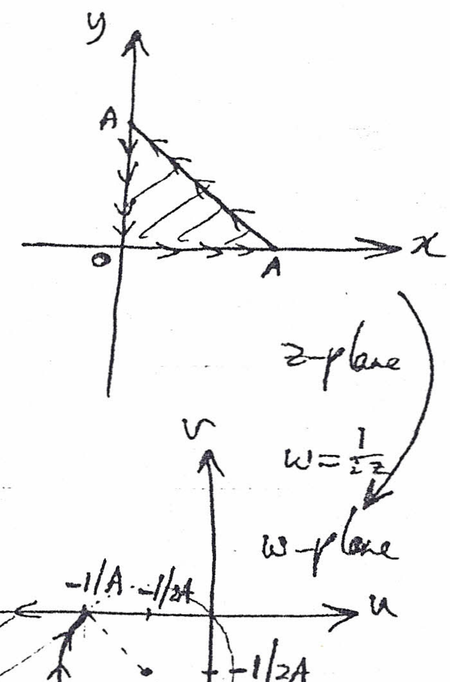
$\therefore \sin c < 0$ & $\cos c < 0$ $\therefore u = \sin c \cosh y \left(= (-ve).(+ve) = -ve\right) < 0$

boundaries are: $x = -\pi \Rightarrow u = 0$ & $x = -\pi/2 \Rightarrow v = 0, u < 0$

Hence, the map as shown above, is a group of hyperbola branches on the negative u plane intersecting it in the interval $(-1,0)$.



$f(z) = \frac{1}{iz} = \frac{1}{-y+ix} = \frac{-y-ix}{x^2+y^2} = u+iv$



$\therefore u = -\frac{y}{x^2+y^2}$ & $v = \frac{-x}{x^2+y^2}$

$\therefore$ Segment along x: $y=0$, $x\in[0,A]$

$\therefore$ Image: $u=0$ & $v\in\left(-\infty, -\frac{1}{A}\right]$

& Segment along y: $x=0$, $y\in[0,A]$

$\therefore$ Image: $v=0$ & $u\in\left(-\infty,-\frac{1}{A}\right]$

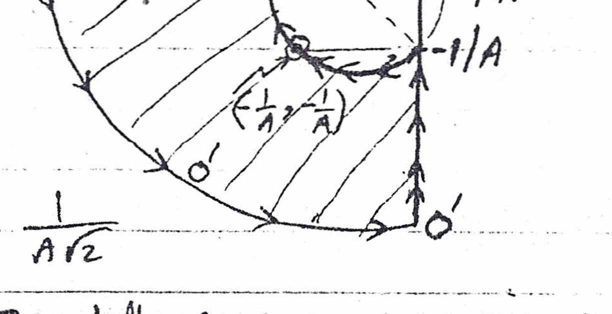
& Segment $x+y=A$ goes to:

$$\frac{x+y}{x^2+y^2} = \frac{A}{x^2+y^2} \Rightarrow$$

$$-v-u = A(u^2+v^2) \quad \text{OR}$$

$$u^2+v^2+\frac{u}{A}+\frac{v}{A}=0 \quad \text{OR}$$

$$\left(u+\frac{1}{2A}\right)^2+\left(v+\frac{1}{2A}\right)^2 = \left(\frac{1}{A\sqrt{2}}\right)^2$$



Circle, centre $\left(-\frac{1}{2A}, -\frac{1}{2A}\right)$, radius $\frac{1}{A\sqrt{2}}$

Passing through origin of w-plane.

$\because x, y > 0$ $\therefore u, v < 0$, $\therefore$ One half of the circle at third quadrant.

Hence, the image is obtained as shown above.

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$\boxed{\frac{8}{100}}$

a) $w = \sin z = \sin x \cosh y + i \cos x \sinh y$

$\therefore u = \sin x \cosh y$ & $v = \cos x \sinh y$

$\therefore$ AB: $x = \frac{\pi}{2}, y \in [0, \infty) \Rightarrow v = 0, u \in [1, \infty)$

& BC: $y = 0, x \in [0, \frac{\pi}{2}] \Rightarrow v = 0, u \in [0, 1]$

& CD: $x = 0, y \in [0, \infty) \Rightarrow u = 0, v \in [0, \infty)$

b) $w = \cos z = \cos x \cosh y - i \sin x \sinh y$

$\therefore u = \cos x \cosh y$ & $v = -\sin x \sinh y$

$\therefore$ AB: $x = \frac{\pi}{2}, y \in [0, \infty) \Rightarrow u = 0$ & $v \in (-\infty, 0]$

& BC: $y = 0, x \in [0, \frac{\pi}{2}] \Rightarrow v = 0$ & $u \in [0, 1]$

& CD: $x = 0, y \in [0, \infty) \Rightarrow v = 0$ & $u \in [1, \infty)$

c) $w = e^z = e^{x+iy} = e^x \cdot e^{iy} =$

$= e^x (\cos y + i \sin y) = u + iv$

$\therefore u(x, y) = e^x \cos y$

& $v(x, y) = e^x \sin y$

$\therefore$ AB: $x = \frac{\pi}{2}, y \in [0, \infty) \Rightarrow |w| = e^{\pi/2}$,

$\angle w \in [0, 2\pi]$ infinite no. of cycles

& BC: $y = 0, x \in [0, \frac{\pi}{2}] \Rightarrow v = 0$ & $u \in [1, e^{\pi/2}]$

& CD: $x = 0, y \in [0, \infty) \Rightarrow |w| = 1$,

$\angle w \in [0, 2\pi]$ infinite no. of cycles

d) $w = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = u + iv$

$\therefore u(x, y) = \frac{x}{x^2 + y^2}$, $v(x, y) = \frac{-y}{x^2 + y^2}$

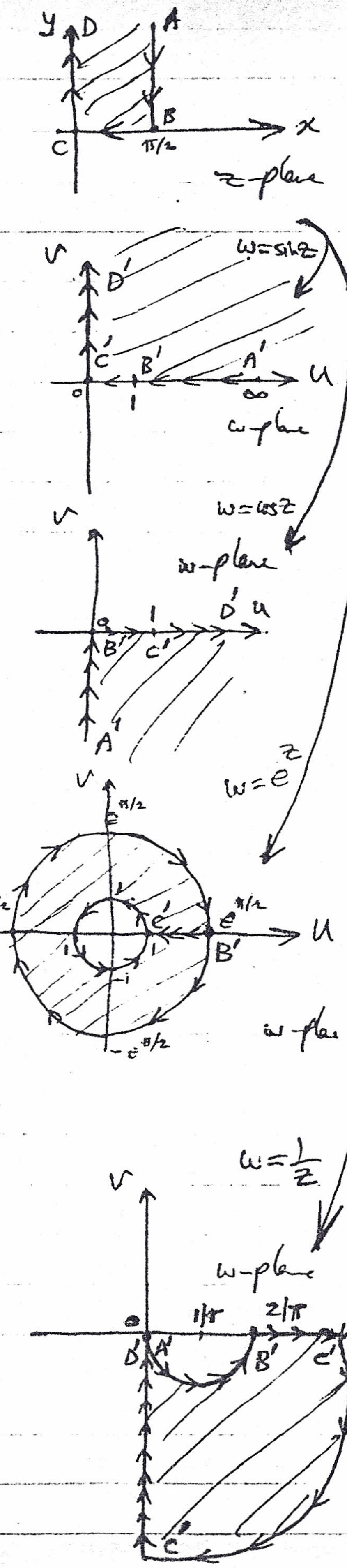
& $x(u, v) = \frac{u}{u^2 + v^2}$, $y(u, v) = \frac{-v}{u^2 + v^2}$

$\therefore$ AB: $x = \frac{\pi}{2}, y \in [0, \infty) \Rightarrow \frac{\pi}{2} = \frac{u}{u^2 + v^2}, v < 0$

$\therefore u^2 + v^2 - \frac{2}{\pi} u = 0 \Rightarrow (u - \frac{1}{\pi})^2 + v^2 = \frac{1}{\pi^2}$, half circle

& BC: $y = 0, x \in [0, \frac{\pi}{2}] \Rightarrow v = 0$ & $u \in [\frac{2}{\pi}, \infty)$

& CD: $x = 0, y \in [0, \infty) \Rightarrow u = 0$ & $v \in (-\infty, 0]$



$\frac{10}{10}$ ③ Consider the boundaries $x=0,\ y=0,\ x=\frac{\pi}{2},\ y=\frac{\pi}{2}$

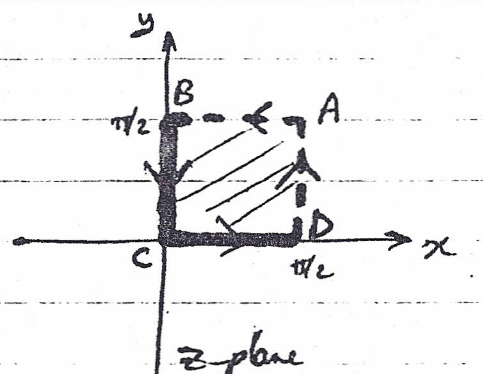
ⓐ when $w=e^z$

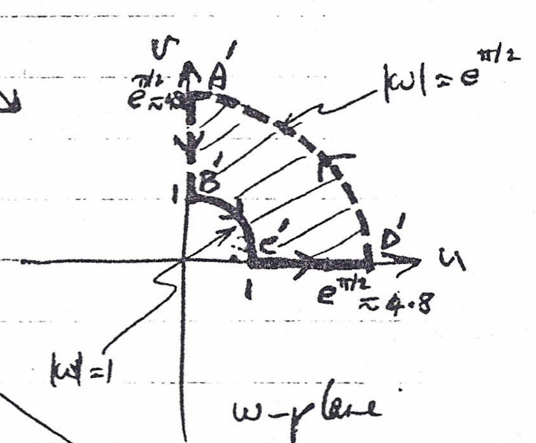
$\therefore$ $x=0$ maps to $e^{0+iy}=e^{iy}$, $\because y\in[0,\frac{\pi}{2}]$ $\therefore$ it is $\frac{1}{4}$ unit circle (1st quarter)

& $y=0$ maps to $e^{x+i0}=e^{x}$, $\therefore u=e^x, v=0$ & $u\in$ (for $x\in[0,\pi/2]$) $[1, e^{\pi/2}]$

& $x=\pi/2$ maps to $e^{\pi/2+iy}=e^{\pi/2}\cdot e^{iy}$, $\because y\in[0,\pi/2]$ $\therefore$ it is $\frac{1}{4}$, $e^{\pi/2}$ circle (1st quarter)

& $y=\pi/2$ maps to $e^{x+i\pi/2}=e^{x}(\cos\pi/2+i\sin\pi/2)=ie^{x}$ $\therefore u=0,\ v=e^{x}\in$ (for $x\in[0,\pi/2]$) $[1, e^{\pi/2}]$





OR ⓑ when $w=\sin z$

$\therefore$ $x=0$ maps to $w=\sin iy=i\sinh y$

$\therefore u=0$ & v (for $y\in[0,\pi/2]$) $\in[0,\sinh\pi/2]$

& $y=0$ maps to $w=\sin x$

$\therefore v=0$ & u (for $x\in[0,\pi/2]$) $\in[0,1]$

& $x=\pi/2$ maps to $w=\sin(\pi/2+iy)=$

$=\cos iy=\cosh y$ $\therefore v=0$ & u (for $y\in[0,\pi/2]$) $\in[1,\cosh\pi/2]$

& $y=\pi/2$ maps to $w=\sin(x+i\pi/2)=$

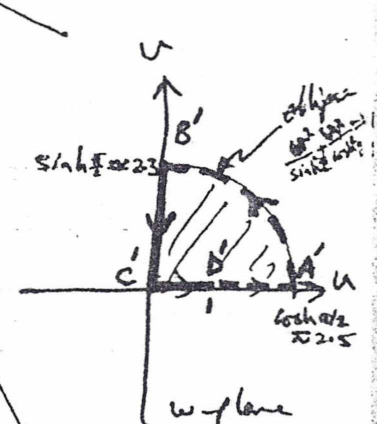
$=\sin x\cos i\pi/2+\cos x\sin i\pi/2=$

$=\sin x\cosh\pi/2+i\cos x\sinh\pi/2=u+iv$

$\therefore \left(\frac{u}{\cosh\pi/2}\right)^2+\left(\frac{v}{\sinh\pi/2}\right)^2=\sin^2x+\cos^2x=1$

$\therefore$ $y=\pi/2$ maps to ellipse $\left(\frac{u}{\cosh\pi/2}\right)^2+\frac{v}{(\sinh\pi/2)^2}=1$

$\because x\in[0,\pi/2]$ $\therefore$ it is in the first quarter.



OR ⓒ when $w=\cos z$

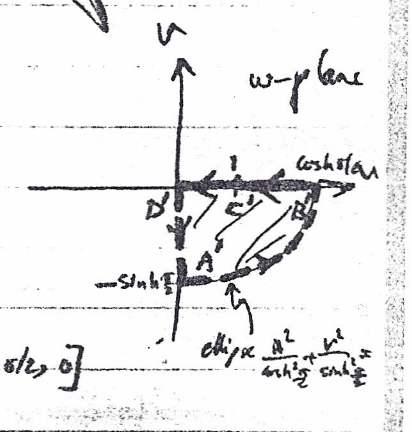
$\therefore$ $x=0$ maps to $w=\cos iy=\cosh y$ $\therefore v=0,\ u\in[1,\cosh\pi/2]$

& $y=0$ " $w=\cos x$ $\therefore v=0,\ u\in[0,1]$

& $x=\pi/2$ " $w=\cos(\pi/2+iy)=-\sin iy=-i\sinh y$ $\therefore u=0,\ v\in[-\sinh\pi/2, 0]$

& $y=\pi/2$ " $w=\cos(x+i\pi/2)=\cos x\cosh\pi/2-i\sin x\sinh\pi/2=u+iv$

$\therefore \left(\frac{u}{\cosh\pi/2}\right)^2+\left(\frac{v}{\sinh\pi/2}\right)^2=\cos^2x+\sin^2x=1$, u +ve, v −ve $\therefore$ $\frac{1}{4}$ ellipse in fourth quadrant.



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(7) $\omega = (\sin z)^{\frac{1}{4}}$ Let $\omega_1 = \sin z$ $\therefore$ $\omega = \omega_1^{1/4}$ (two transformations)

Now $x \in [-\pi/2, \pi/2]$, $y \in [0, \infty)$.

From the analysis above $x = c$, $c \in (-\pi/2, 0)$ fourth quadrant, $y \geq 0$, will now map to hyperbola $\frac{1}{2}$ branches in ω_1 second quadrant because $\begin{cases}\sin c < 0\\ \cos c > 0\end{cases}$ & $y \geq 0$.

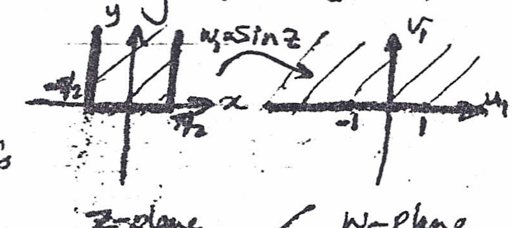
Similarly, $x = c$, $c \in (0, \pi/2)$ first quadrant, $y \geq 0$, will map to hyperbolas $\frac{1}{2}$ branches in ω_1 first quadrant because $\sin c$, $\cos c$ & y are all > 0 or $y = 0$.

$\therefore$ $x = c$, $y \geq 0$, $c \in (-\pi/2, \pi/2)$, $c \neq 0$ will map into upper $\frac{1}{2}$ hyperbolas intersecting u_1 in $(-1, 1)$ $u_1 \neq 0$

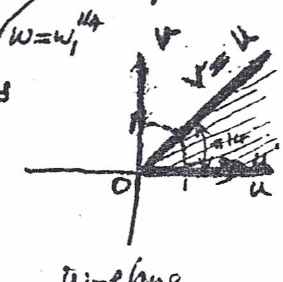
& if we put $c = 0$ then the immage is $w_1 = \sin iy = i \sinh y$, i.e. $u_1 = 0$

$\therefore$ $x = c \in (-\pi/2, \pi/2)$, $y \geq 0$ maps into upper $\frac{1}{2}$ hyperbolas intersecting u_1 in $(-1, 1)$

Hence, the transformation $x \in [-\pi/2, \pi/2]$ will map to the upper w_1 plane where boundary lines $x = \pm \pi/2$, $y \geq 0$ maps to segments on u_1-axis where: $u_1 \in \pm [1, \infty)$, $u_1 \in [-1, 1]$ respectively



Applying the transformation $\omega = w_1^{1/4}$, to w_1 - upper half (one sheet) will result in mapping the w_1 ($\frac{1}{2}$ plane) into the first half quadrant of the w-plane ($\angle w = \angle w_1/4$) and the boundary $v_1 = 0$, $u_1 \in [-1, \infty)$ will map to $\angle w = \frac{\pi}{4}$, $|w| \in [1, \infty)$.



The boundary $v_1 = 0$, $u_1 \in [1, \infty)$ will map to line $\angle w = 0$, $|w| \in [1, \infty)$, and

Finally, the boundary $v_1 = 0$, $u_1 \in [-1, 1]$ will map to extension of above lines to 0 as shown by solid lines.

صائم

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(4) $w = z + \frac{1}{z}$, $z = re^{i\theta}$

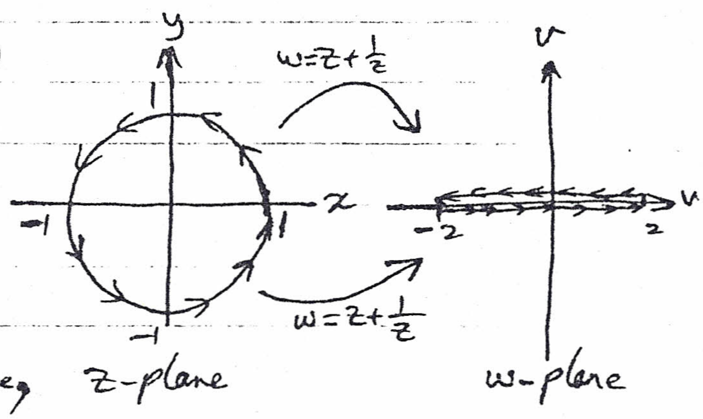
$\therefore w(z) = re^{i\theta} + \frac{1}{re^{i\theta}}$ when $r=1$

$\therefore w(z) = e^{i\theta} + e^{-i\theta} = 2\cos(\theta) = u + iv$ $\therefore v=0$, $u = 2\cos\theta$

$\therefore$ $r=1$, $\theta \in [-\pi, 0]$ (lower half circle) will map to segment on u-axis given by $u \in [-2, 2]$

& $r=1$, $\theta \in [0, \pi]$ (upper half circle) will map to segment on u-axis given by $u \in [2, -2]$ as shown.

$\therefore$ Both upper and lower halves of unit circle $r=1$ are mapped onto the u-axis between $[-2,2]$, and going around the circle from the point $(-1,0)$ to $(-1,0)$ in anti clockwise direction in z-plane, results in going from the point $(-2,0)$ straight to $(2,0)$ and comming back from $(2,0)$ to $(-2,0)$ in w-plane.



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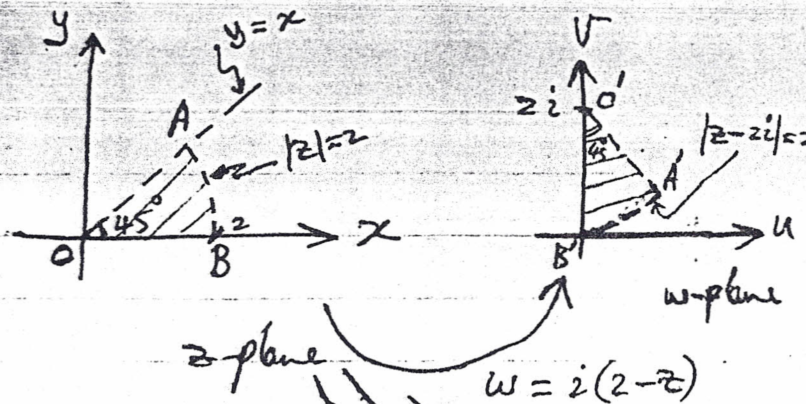
$$w = i(2-z) = -i(z-2)$$

$\therefore$ Shift by 2 to left then rotate clockwise by 90°.



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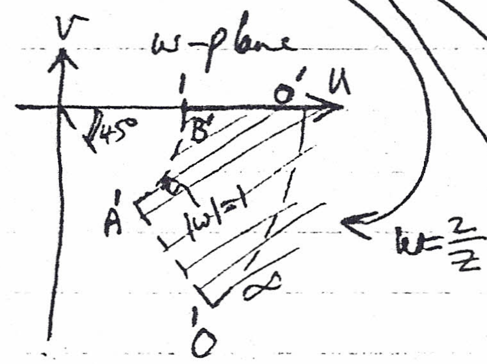
$$W = \frac{2}{z} = 2 \cdot \frac{x - iy}{x^2+y^2}$$

$\therefore\ u = \dfrac{2x}{x^2+y^2}$

& $v = \dfrac{-2y}{x^2+y^2}$



$\therefore$ OB: $y=0,\ x \in [0,2]\ \therefore\ v=0$ & $u \in [1,\infty)$

& BA: $|z|=2\ \angle z \in [0,45^\circ)\ \therefore\ |w| = \frac{2}{2} = 1$ & $\angle w = (-45^\circ, 0]$

& AO: $y = x\ \therefore\ \dfrac{u}{v} = \dfrac{x}{-y} = -1\ \therefore\ u = -v$

#

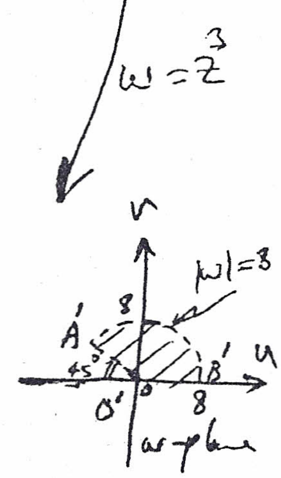
$$w = z^3$$

$\therefore\ |w| = |z|^3$ & $\angle w = 3\angle z$

$\therefore$ OB: $\angle z = 0$ & $|z| \in [0,2]\ \therefore\ \angle w = 0, |w| \in [0,8]$

& BA: $|z| = 2, \angle z \in [0, 45^\circ)\ \therefore |w| = 8, \angle w \in [0, 135^\circ)$

& AO: $\angle z = 45^\circ, |z| \in (2, 0]\ \therefore\ \angle w = 135^\circ, |w| \in (8, 0]$



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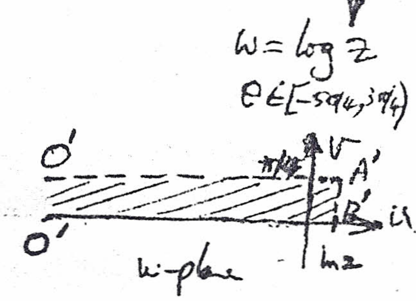
$$w = \log z \quad , \quad \theta \in \left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right) = [-225^\circ, 135^\circ)$$

$\therefore\ w = \log|z| + i\angle z\ \therefore\ u = \ln|z|,\ v = \angle z = \theta$

$\therefore$ OB: $\theta = 0,\ |z| \in [0,2]\ \therefore\ v = 0,\ u \in (-\infty, \ln 2]$

& BA: $|z| = 2,\ \theta \in [0, 45^\circ)\ \therefore\ u = \ln 2,\ v \in [0, \frac{\pi}{4})$

& AO: $\theta = 45^\circ,\ |z| \in (2, 0]\ \therefore\ v = \frac{\pi}{4},\ u \in (\ln 2, -\infty)$



$w = e^z = e^x(\cos y + i\sin y) = e^x \angle y$

∴ $u = e^x \cos y$

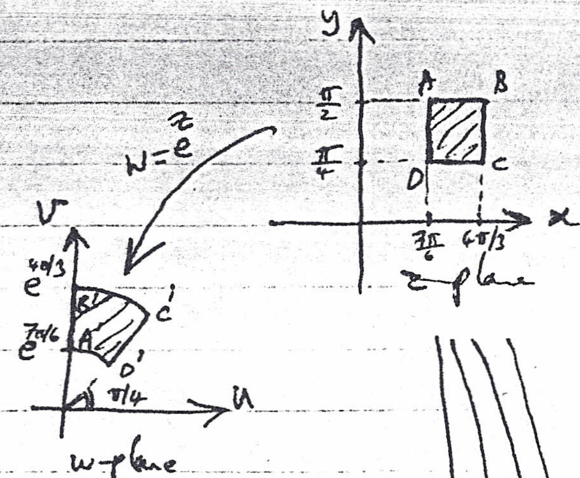
& $v = e^x \sin y$

∴ AB: $y = \frac{\pi}{2}$ ∴ $u = 0$ & $v = e^x \in [e^{7\pi/6}, e^{4\pi/3}]$

& BC: $x = \frac{4\pi}{3}$ ∴ $|w| = e^{4\pi/3}$, $\angle w \in [\frac{\pi}{2}, \frac{\pi}{4}]$

& CD: $y = \frac{\pi}{4}$ ∴ $\angle w = \frac{\pi}{4}$, $|w| \in [e^{4\pi/3}, e^{7\pi/6}]$

& DA: $x = \frac{7\pi}{6}$ ∴ $|w| = e^{7\pi/6}$, $\angle w \in [\frac{\pi}{4}, \frac{\pi}{2}]$



$w = \sin z = \sin x \cosh y + i \cos x \sinh y$

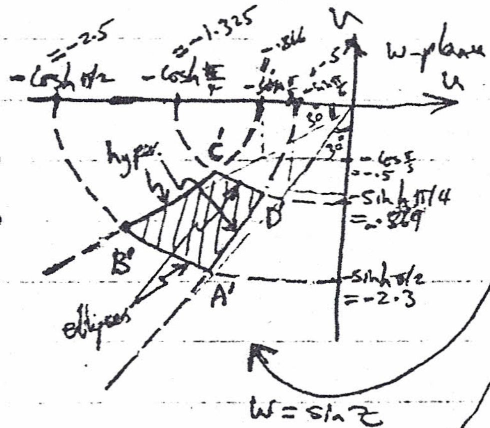
$\frac{1}{4}$ ∴ $u = \sin x \cosh y$ & $v = \cos x \sinh y$

∴ AB: $y = \frac{\pi}{2}$ ∴ $\left(\frac{u}{\cosh \pi/2}\right)^2 + \left(\frac{v}{\sinh \pi/2}\right)^2 = 1$, u & $v < 0$

& BC: $x = \frac{4\pi}{3}$ ∴ $\left(\frac{u}{\sin 4\pi/3}\right)^2 - \left(\frac{v}{\cos 4\pi/3}\right)^2 = 1$, u & $v < 0$

& CD: $y = \frac{\pi}{4}$ ∴ $\left(\frac{u}{\cosh \pi/4}\right)^2 + \left(\frac{v}{\sinh \pi/4}\right)^2 = 1$, u & $v < 0$

& DA: $x = \frac{7\pi}{6}$ ∴ $\left(\frac{u}{\sin 7\pi/6}\right)^2 - \left(\frac{v}{\cos 7\pi/6}\right)^2 = 1$, u & $v < 0$



$w = \cos z = \cos x \cosh y - i \sin x \sinh y$

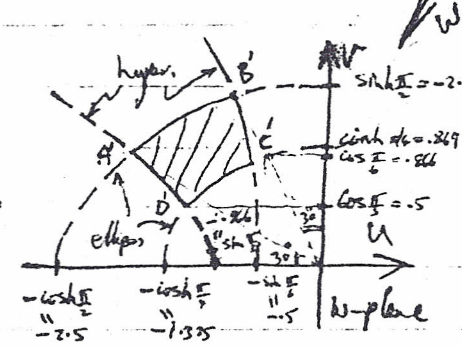
∴ $u = \cos x \cosh y$ & $v = -\sin x \sinh y$

∴ AB: $y = \frac{\pi}{2}$ ∴ $\left(\frac{u}{\cosh\frac{\pi}{2}}\right)^2 + \left(\frac{v}{\sinh \pi/2}\right)^2 = 1$, $u < 0, v > 0$

& BC: $x = \frac{4\pi}{3}$ ∴ $\left(\frac{u}{\cos 4\pi/3}\right)^2 - \left(\frac{v}{\sin 4\pi/3}\right)^2 = 1$, $u < 0, v > 0$

& CD: $y = \frac{\pi}{4}$ ∴ $\left(\frac{u}{\cosh \pi/4}\right)^2 + \left(\frac{v}{\sinh \pi/4}\right)^2 = 1$, $u < 0, v > 0$

& DA: $x = \frac{7\pi}{6}$ ∴ $\left(\frac{u}{\cos 7\pi/6}\right)^2 + \left(\frac{v}{\sin 7\pi/6}\right)^2 = 1$, $u < 0, v > 0$



$w = \sinh z = \sinh x \cos y + i \cosh x \sin y$

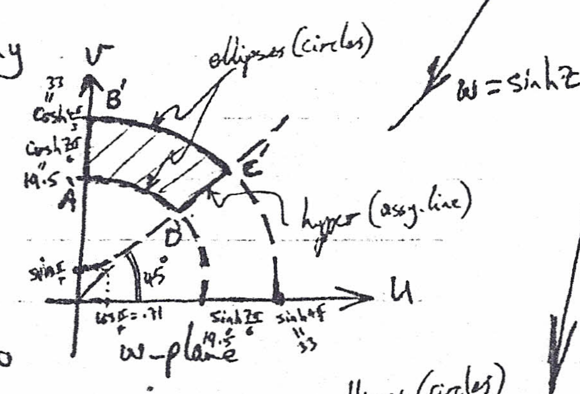
$\frac{1}{4}$ ∴ $u = \sinh x \cos y$ & $v = \cosh x \sin y$

∴ AB: $y = \frac{\pi}{2}$ ∴ $u = 0$, $v \in [\cosh\frac{7\pi}{6}, \cosh\frac{4\pi}{3}]$

& BC: $x = \frac{4\pi}{3}$ ∴ $\left(\frac{u}{\sinh 4\pi/3}\right)^2 + \left(\frac{v}{\cosh\frac{4\pi}{3}}\right)^2 = 1$, u & $v > 0$

& CD: $y = \frac{\pi}{4}$ ∴ $\left(\frac{u}{\cos \pi/4}\right)^2 + \left(\frac{v}{\sin \pi/4}\right)^2 = 1$, u & $v > 0$

& DA: $x = \frac{7\pi}{6}$ ∴ $\left(\frac{u}{\sinh 7\pi/6}\right)^2 + \left(\frac{v}{\cosh 7\pi/6}\right)^2 = 1$, u & $v > 0$



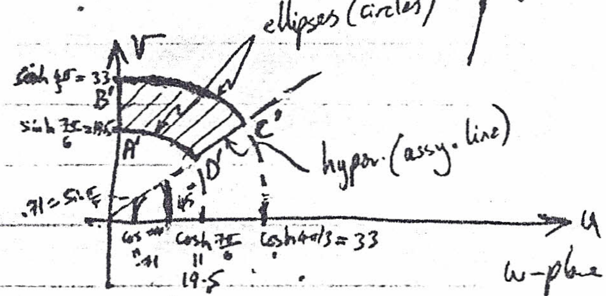
$w = \cosh z = \cosh x \cos y + i \sinh x \sin y$

∴ $u = \cosh x \cos y$ & $v = \sinh x \sin y$

∴ AB: $y = \frac{\pi}{2}$ ∴ $u = 0$ & $v \in [\sinh\frac{7\pi}{6}, \sinh\frac{4\pi}{3}]$

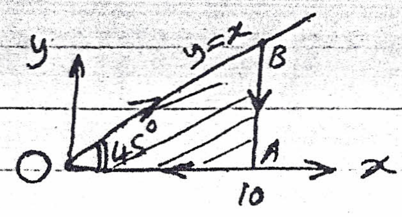
& BC: $x = \frac{4\pi}{3}$ ∴ $\left(\frac{u}{\cosh\frac{4\pi}{3}}\right)^2 + \left(\frac{v}{\sinh\frac{4\pi}{3}}\right)^2 = 1$, u & $v > 0$

& CD: $y = \frac{\pi}{4}$ ∴ $\left(\frac{u}{\cos \pi/4}\right)^2 - \left(\frac{v}{\sin \pi/4}\right)^2 = 1$, u & $v > 0$; & DA: $x = \frac{7\pi}{6}$ ∴ $\left(\frac{u}{\cosh\frac{7\pi}{6}}\right)^2 + \left(\frac{v}{\sinh\frac{7\pi}{6}}\right)^2 = 1$, u & $v > 0$



$w = \frac{i}{z}$

Let $w_1 = 1/z$



∴ $w = i w_1$

First $w_1 = 1/z$

section OB $|z| \in [0, 10\sqrt{2}]$ & $\angle z = 45°$.

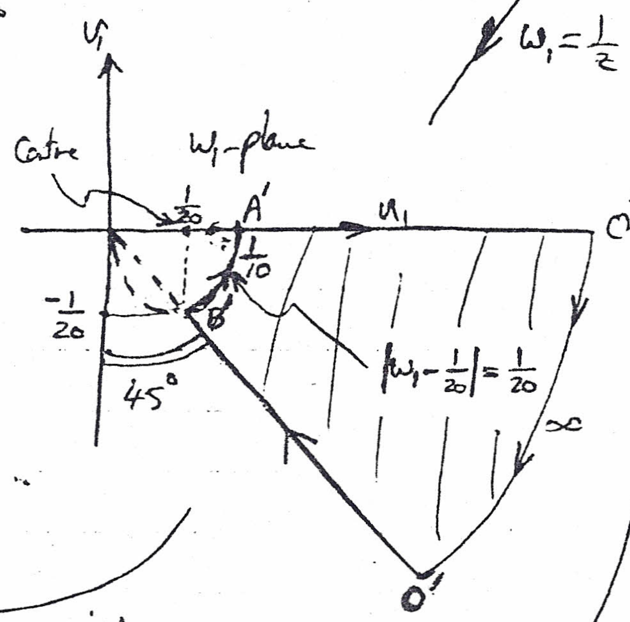
∴ $O'B'$ is $|w_1| = \frac{1}{|z|} \in [\infty, \frac{1}{10\sqrt{2}}]$ & $\angle w_1 = -\angle z = -45°$, straight line

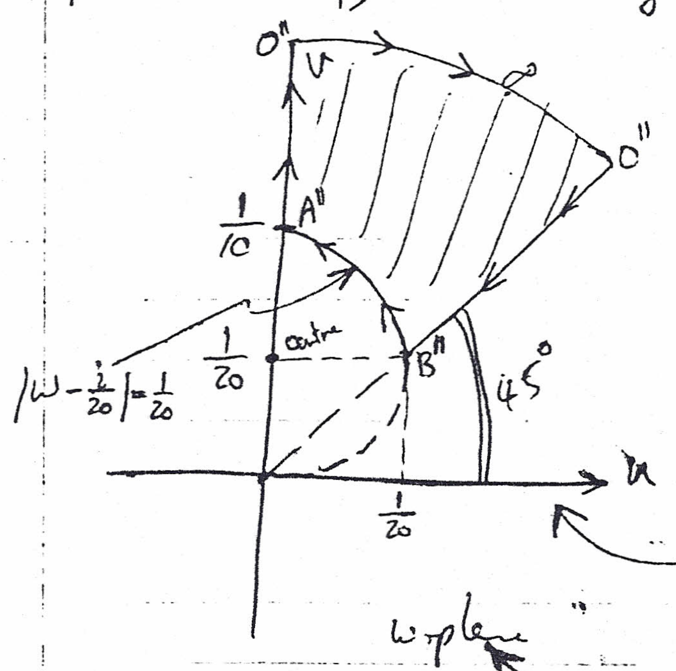
Section BA is a line not through O // y-axis

∴ $B'A'$ is $\frac{1}{8}$ circle through O centre at $\frac{1}{20}$ on u_1-axis

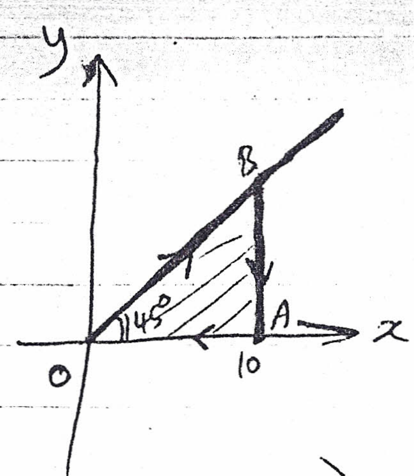
section AO is mapped into u_1 axis $[\frac{1}{10}, \infty)$

Next $w = i w_1$, i.e. rotation of 90°





$w = z^2 = x^2 - y^2 + i\,2xy$
$\therefore\ u = x^2 - y^2$
& $v = 2xy$

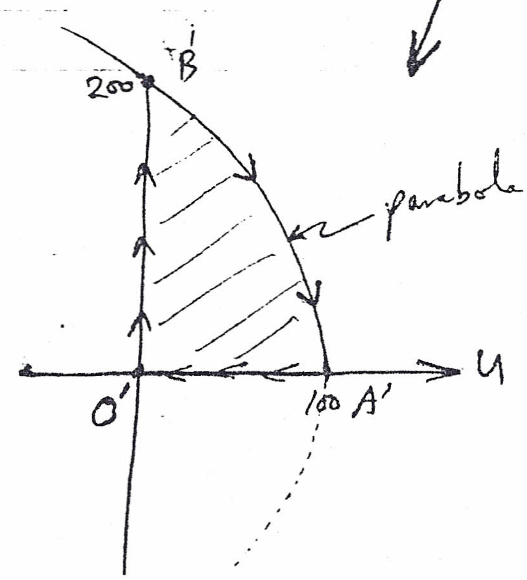


OB: $y = x$, $x \in [0, 10]$
$\therefore\ u = x^2 - x^2 = 0$
& $v = 2x^2 > 0$, $\in [0, 2(10^2)] = [0, 200]$

BA: $x = 10$, $y \in [+10, 0]$
$\therefore\ u = 100 - y^2$
& $v = 20y > 0$
$\therefore\ u = 100 - \left(\frac{v}{20}\right)^2$, parabola, vertex $(100, 0)$
axis along u, open to left, intercept of v at ± 200

AO: $y = 0$, $x \in [10, 0]$
$\therefore\ v = 0$ & $u = x^2 > 0$, $\in [100, 0]$

Hence, we get the shown mapping.



$\boxed{\frac{1a}{106}}$

$$\int_0^{\pi/4} e^{it}\,dt = \int_0^{\pi/4} (\cos t + i \sin t)\,dt = \int_0^{\pi/4} \cos t\,dt + i\int_0^{\pi/4} \sin t\,dt =$$

$$= \sin t\Big]_0^{\pi/4} + i\left(\frac{-\cos t}{1}\right)\Big|_0^{\pi/4} = \sin(\pi/4) - i\left(\cos(\pi/4) - 1\right) =$$

$$= \sin(\pi/4) - i\cos(\pi/4) + i = \frac{1-i}{\sqrt{2}} + i = \frac{1 + (\sqrt{2}-1)i}{\sqrt{2}}$$

(OR, directly $\int_0^{\pi/4} e^{it}\,dt = \frac{e^{it}}{i}\Big|_0^{\pi/4} = \frac{e^{i\pi/4} - e^{0}}{i} = \frac{\cos 45° + i \sin 45° - 1}{i} =$

$= \frac{1}{i\sqrt{2}}(1 + i - \sqrt{2}) = \frac{-i}{\sqrt{2}}(1 + i - \sqrt{2}) = \frac{1 + (\sqrt{2}-1)i}{\sqrt{2}}$, $\therefore$ OK)

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HW15 (27)

بسم الله الرحمن الرحيم

حل مسائل الواجب المطلوب يوم الأربعاء ٢٧/٤/١٤٠٨هـ

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$$\int_0^\infty e^{-zt}\,dt \quad (\operatorname{Re} z>0) \quad , \text{ Let } z = x+iy \quad \operatorname{Re} z>0 \ \therefore x>0$$

$$\therefore z = \sqrt{x^2}+iy$$

$$\therefore \int_0^\infty e^{-zt}\,dt = \int_0^\infty e^{-(\sqrt{x^2}+iy)t}\,dt =$$

$$= \int_0^\infty e^{-\sqrt{x^2}t}\cdot e^{-iyt}\,dt = \int_0^\infty e^{-\sqrt{x^2}t}\cdot(\cos yt - i\sin yt)\,dt =$$

$$= \int_0^\infty e^{-\sqrt{x^2}t}\cos yt\,dt - i\int_0^\infty e^{-\sqrt{x^2}t}\sin yt\,dt = I_1 - iI_2$$

$$\therefore I_1 = \int_0^\infty e^{-\sqrt{x^2}t}\cos yt\,dt = e^{-\sqrt{x^2}t}\cdot\frac{\sin yt}{y}\Big]_0^\infty - \int_0^\infty \frac{\sin yt}{y}\cdot(-\sqrt{x^2})\,e^{-\sqrt{x^2}t}\,dt =$$

$$= e^{-\infty}\cdot\frac{\sin\infty y}{y}(=0) - e^{0}\cdot\frac{\sin 0}{y} + \frac{\sqrt{x^2}}{y}\int_0^\infty \sin yt\, e^{-\sqrt{x^2}t}\,dt =$$

$$= \frac{\sqrt{x^2}}{y}\left[e^{-\sqrt{x^2}t}\cdot\left(\frac{-\cos yt}{y}\right)\Big|_0^\infty - \int_0^\infty\left(\frac{-\cos yt}{y}\right)\cdot(-\sqrt{x^2})\,e^{-\sqrt{x^2}t}\,dt\right] =$$

$$= \frac{\sqrt{x^2}}{y}\cdot\left[-\frac{e^{-\infty}\cos y\infty}{y}(=0) + \frac{e^{0}\cos 0}{y} - \frac{\sqrt{x^2}}{y}\,I_1\right] = \frac{\sqrt{x^2}}{y}\cdot\frac{1}{y} - \frac{x^2}{y^2}\cdot I_1$$

3 $$\therefore I_1\left(1+\frac{x^2}{y^2}\right) = \frac{\sqrt{x^2}}{y^2} \qquad \therefore I_1 = \frac{\sqrt{x^2}}{x^2+y^2} = \frac{x}{x^2+y^2} \quad (x>0)$$

$$\text{Also, } I_2 = \int_0^\infty e^{-\sqrt{x^2}t}\sin yt\,dt = -\frac{1}{y}\int_{t=0}^\infty e^{-\sqrt{x^2}t}\,d\cos yt = \frac{-e^{-\sqrt{x^2}t}\cos yt}{y}\Big|_0^\infty - \frac{\sqrt{x^2}}{y}\int_0^\infty e^{-\sqrt{x^2}t}\cos yt\,dt =$$

$$= 0 + \frac{1}{y} - \frac{\sqrt{x^2}}{y}\left[e^{-\sqrt{x^2}t}\cdot\frac{\sin yt}{y}\Big|_0^\infty - \int\frac{\sin yt}{y}\cdot e^{-\sqrt{x^2}t}(-\sqrt{x^2})\,dt\right] = \frac{1}{y} - \frac{\sqrt{x^2}}{y}\left[0-0+\frac{\sqrt{x^2}}{y}\cdot I_2\right]$$

3 $$\therefore I_2\left(1+\frac{x^2}{y^2}\right) = \frac{1}{y} \qquad \therefore I_2 = \frac{y}{y^2+x^2}$$

$$\therefore \int_0^\infty e^{-zt}\,dt = I_1 - iI_2 = \frac{x}{x^2+y^2} - i\cdot\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{\bar z}{|z|^2} = \frac{\bar z}{z\bar z} = \frac{1}{z}$$

1 $$\therefore \int_0^\infty e^{-zt}\,dt = \frac{1}{z} \quad , \operatorname{Re} z>0$$

$\frac{3}{107}$ ④ $\int_0^{2\pi} e^{imt} \cdot e^{-int}\, dt = \int_0^{2\pi} e^{i(m-n)t}\, dt = \int_0^{2\pi} \left[\cos(m-n)t + i\sin(m-n)t\right] dt$

$= \int_0^{2\pi} \cos(m-n)\,t\, dt + i \int_0^{2\pi} \sin(m-n)t\, dt = I_1 + i\, I_2$. Let $m-n=k$

$\therefore\ I_1 = \int_0^{2\pi} \cos(m-n)\,t\, dt = \int_0^{2\pi} \cos kt\, dt =$

$$= \begin{cases} k=0 & \int_0^{2\pi} \cos 0\, dt = \int_0^{2\pi} dt = t\Big]_0^{2\pi} = 2\pi \\ k\neq 0 & \int_0^{2\pi} \cos kt\, dt = \frac{\sin kt}{k}\Big]_0^{2\pi} = \frac{0-0}{k}\ (,\ k\neq 0) = 0 \end{cases}$$

& $I_2 = \int_0^{2\pi} \sin(m-n)\, t\, dt = \int_0^{2\pi} \sin kt\, dt =$

$$= \begin{cases} k=0 & \int_0^{2\pi} \sin 0\, dt = \int 0\, dt = 0 \\ k\neq 0 & \int_0^{2\pi} \sin kt\, dt = -\frac{\cos kt}{k}\Big]_0^{2\pi} = \frac{1-1}{-k}\ (,\ k\neq 0) = 0 \end{cases}$$

$$\therefore\ I_1 + i\, I_2 = \begin{cases} k=0 & 2\pi + i0 = 2\pi \\ k\neq 0 & 0 + i0 = 0 \end{cases}$$

(note: $k=0 \Rightarrow m=n$)

$\therefore \int_0^{2\pi} e^{imt} \cdot e^{-int}\, dt = 2\pi$ when $m=n$ & $=0$ otherwise

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$\frac{4}{113}$ $\int_{c} f(z)\,dz$ $c: y = x^3,\ x\in[-1,1]$ $= \int_{\substack{c\\ y<0}} 1\, dz + \int_{\substack{c\\ y>0}} 4y\, dz = \int_{\substack{c\\ y<0}} (dx + i\,dy) + \int_{\substack{c\\ y>0}} 4y\,(dx + i\,dy)$

$= \int_{\substack{c\\ y<0}} (1 + iy')\, dx + \int_{\substack{c\\ y>0}} 4y\,(1+iy')\, dx = \int_{-1}^{0} (1 + i3x^2)\, dx + \int_0^1 4x^3(1+i3x^2)\, dx$

$= \left[x + ix^3\right]_{-1}^{0} + \left[x^4 + i2x^6\right]_0^1 = 0 + (1+i) + (1+2i) - 0 = 2 + 3i$

$\frac{6}{113}$ ④ $\int_c z^m \cdot \bar{z}^n \, dz$, m & n are integers & c is $|z|=1$ counterclockwise

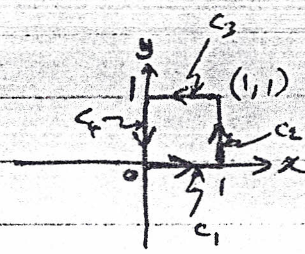
$\therefore z = |z| \cdot e^{i\theta}\Big|_{at\,c} = 1 \cdot e^{i\theta} = e^{i\theta} \quad \therefore z^m = e^{im\theta}$

& $\bar{z} = |z| \cdot e^{-i\theta}\Big|_{at\,c} = 1 \cdot e^{-i\theta} = e^{-i\theta} \quad \therefore \bar{z}^n = e^{-in\theta}$

$dz = d\,e^{i\theta} = i e^{i\theta} \, d\theta$, θ changes from 0 to 2π (counterclockwise)

$\therefore \int_c z^m \cdot \bar{z}^n dz = \int e^{im\theta} \cdot e^{-in\theta} \cdot i e^{i\theta} d\theta = i \int e^{i(m+1-n)\theta} d\theta = i * \begin{cases} 2\pi & m+1-n=0 \\ 0 & \neq 0 \end{cases}$ (see $\frac{3}{107}$) $= \begin{cases} 2\pi i & (\text{at } n = m+1) \\ 0 & (\text{at } n \neq m+1) \end{cases}$

$\frac{8}{114}$ (2) $I=\oint_C \pi e^{\pi\bar z}dz = \oint_{C_1+C_2+C_3+C_4}\pi e^{\pi \bar z}dz =$



$$=\left(\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right)\pi e^{\pi\bar z}dz =$$

$$\therefore \int_{C_1}\pi e^{\pi\bar z}dz = \int_{\substack{C_1:\ y=0,\ x\in[0,1]\\ \therefore dy=0}} \pi e^{\pi(x-iy)}(dx+idy) = \int_0^1 \pi e^{\pi x}dx = \pi \frac{e^{\pi x}}{\pi}\Big|_0^1 = e^{\pi x}\Big|_0^1 = e^\pi -1$$

$$\& \int_{C_2}\pi e^{\pi\bar z}dz = \int_{\substack{C_2:\ x=1,\ y\in[0,1]\\ \therefore dx=0}} \pi e^{\pi(x-iy)}(dx+idy) = \int_0^1 \pi e^{\pi(1-iy)} \cdot i\,dy = \pi i \times \frac{e^{\pi(1-iy)}}{-\pi i}\Big|_0^1$$

$$= -e^{\pi(1-iy)}\Big|_0^1 = e^{\pi(1-iy)}\Big|_1^0 = e^\pi - e^{\pi(1-i)} = e^\pi(1-e^{-\pi i}) = e^\pi(1-(-1)) = 2e^\pi$$

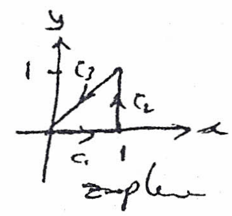
$$\& \int_{C_3}\pi e^{\pi\bar z}dz = \int_{\substack{C_3:\ y=1,\ x\in[1,0]\\ \therefore dy=0}} \pi e^{\pi(x-iy)}(dx+idy) = \int_1^0 \pi e^{\pi(x-i)}dx = e^{\pi(x-i)}\Big|_1^0 = e^{-\pi i} - e^{\pi(1-i)} = e^{-\pi i}(1-e^\pi) = -(1-e^\pi)$$
$$= e^\pi -1$$

$$\& \int_{C_4}\pi e^{\pi\bar z}dz = \int_{\substack{C_4:\ x=0,\ y\in[1,0]\\ \therefore dx=0}} \pi e^{\pi(x-iy)}(dx+idy) = \int_1^0 \pi e^{-\pi i y}\, i\,dy = -e^{-\pi y i}\Big|_1^0 = -1+e^{-\pi i} = -1-1=-2$$

$$\therefore I = \left(\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right)\pi e^{\pi\bar z}dz = e^\pi -1 + 2e^\pi + e^\pi -1 -2 = 4e^\pi - 4 = 4(e^\pi -1)$$

#

$$\oint_C \pi e^{\pi\bar z}dz = \left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right)\pi e^{\pi \bar z}dz$$



$$=(e^\pi-1)+2e^\pi + \int_{C_3}\pi e^{\pi\bar z}dz$$

$$\text{but} \int_{C_3:\ y=x} \pi e^{\pi\bar z}dz = \int \pi e^{\pi(x-iy)}(dx+idy) = \int_1^0 \pi e^{\pi x(1-i)}(1+i)\,dx =$$

$$= \pi \frac{e^{\pi x(1-i)}}{\pi(1-i)}(1+i)\Big|_1^0 = \left(\frac{1+i}{1-i}\right)\left(e^0 - e^{\pi(1-i)}\right) = \frac{(1+i)^2}{1-i^2}\cdot\left(1-e^\pi\cdot e^{-i\pi}\right) =$$

$$= \frac{1-1+2i}{1+1}\left(1-e^\pi\cdot(-1)\right) = i(1+e^\pi)$$

$$\therefore \oint_C \pi e^{\pi\bar z}dz = (e^\pi-1)+2e^\pi + i(1+e^\pi) = 3e^\pi -1 + i(e^\pi+1)$$

$\frac{12}{114}$ $C: |z| = R > 1$ Counter clockwise CCW & $z\big|_{\text{at } c} = Re^{i\theta}$

Let $F(z) = \frac{\text{Log } z}{z^2} = \frac{\ln|z| + i\theta}{|z|^2 \angle 2\theta} = \frac{\ln R + i\theta}{R^2 \angle 2\theta}$, $\theta \in (-\pi, \pi)$ principal branch

$\therefore |F(z)|^2 = \frac{(\ln R)^2 + \theta^2}{R^4}$

now, the maximum of $|F(z)|$ is when θ is max. (R is constant)

$\therefore |F(z)| = \sqrt{\frac{(\ln R)^2 + \theta^2}{R^4}} < \sqrt{\frac{(\ln R)^2 + (\pi)^2}{R^4}} < \frac{(\ln R) + \pi}{R^2} = M$, $R > 1$.

$\therefore \left|\int_C F(z)\,dz\right| < M \cdot \left|\int_C dz\right| = M * L$ (length of contour)

$\because L = 2\pi R$ (contour is a circle radius R)

$\therefore \left|\int_C \frac{\text{Log } z}{z^2}\,dz\right| < 2\pi R \cdot \frac{(\ln R) + \pi}{R^2} = \frac{2\pi}{R}(\pi + \ln R)$

Now $\lim_{R\to\infty} \frac{\ln R}{R}$ ($= \frac{\infty}{\infty}$ indefinite form) $= \lim_{R\to\infty} \frac{(\ln R)'}{(R)'} = \lim_{R\to\infty} \frac{\frac{1}{R}}{1} = \lim_{\frac{1}{R}\to 0} \frac{1}{R} = 0$

$\therefore$ The value of the integral as $R \to \infty$ is bounded by 0^+.

$\therefore$ The integral $\left|\int_C \frac{\text{Log } z}{z^2}\,dz\right|$ approaches zero as $R \to \infty$.

$\left|\oint_C \frac{\text{Log } z}{z^2}\,dz\right| \le \oint_C \frac{|\text{Log } z|\,|dz|}{|z^2|} = \oint_C \frac{|\ln R + i\theta|\,|dz|}{|z|^2}$ $\theta \in [-\pi, \pi)$ $C: |z| = R > 1$

$\le \frac{|\ln R| + |\theta|}{R^2} 2\pi R < \frac{\ln R + \pi}{R} * 2\pi = \frac{2\pi}{R}(\pi + \ln R)$, $\therefore$ OK

$\frac{16}{115}$ ④ Let $\int_{C_0} (z - z_0)^{n-1}\,dz = I_n$, $C_0: |z - z_0| = r_0$, θ CCW

$\therefore z - z_0 = r_0 e^{i\theta}$, $\theta \in [0, 2\pi)$ $\Rightarrow dz = ir_0 e^{i\theta}\,d\theta$

$\therefore I_n = \int_{C_0} (r_0 e^{i\theta_0})^{n-1} \cdot ir_0 e^{i\theta}\,d\theta =$

$= r_0^{n-1} \cdot r_0 \cdot i \int_{C_0} e^{ni\theta}\,d\theta = ir_0^n \int_0^{2\pi} (\cos n\theta + i\sin n\theta)\,d\theta =$ $\begin{cases} n = 0: & ir_0^0 \int_0^{2\pi} 1\,d\theta = i\theta\Big]_0^{2\pi} = 2\pi i \\ n \ne 0: & ir_0^n \left[\frac{\sin n\theta}{n} + i\frac{\cos n\theta}{-n}\right]_0^{2\pi} \end{cases}$

$\therefore I_0$ (I_n when $n = 0$) $= \int_{C_0} (z - z_0)^{0-1}\,dz = \int_{C_0} \frac{dz}{z - z_0} = 2\pi i$

& I_n ($n \ne 0$) $= \int_{C_0} (z - z_0)^{n-1}\,dz = 0$

d/126 (3) $f(z) = \operatorname{sech} z = \frac{1}{\cosh z}$ $\therefore f(z)$ is analytic in its domain of definition, i.e. in all z : $\cosh z \neq 0$

$\because \cosh z = 0 \Rightarrow \cosh x \cos y + i \sinh x \sin y = 0 + i0$

$\therefore \cosh x \cos y = 0 \Rightarrow y = \frac{2n+1}{2}\cdot\pi$

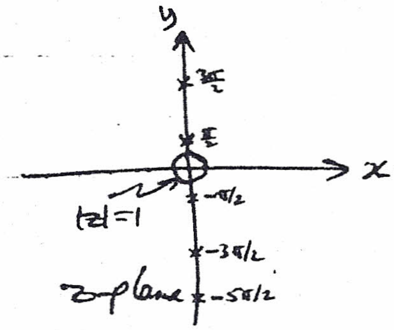
& $\sinh x \sin y = 0 \Rightarrow \sinh x \cdot (\pm 1) = 0 \Rightarrow x = 0$

$\therefore \cosh z = 0 \Rightarrow z = (0, \frac{2n+1}{2}\pi) = \frac{2n+1}{2}\pi i$

$\therefore$ Domain of analyticity = Domain of definition = All z but $z \neq \frac{2n+1}{2}\pi i$, n, integer

The singularities of sech z are shown here:

Now, the contour C is $|z| = 1$ and, from the figure shown, the function is analytic throughout C and its interior.



$\therefore \oint_C f(z)\, dz = 0$ by CGT (Cauchy-Goursat Theorem)

$\therefore \oint_{C\,|z|=1} \operatorname{sech} z\, dz = 0$

7e/126 (2) $\tan z = \frac{\sin z}{\cos z}$ $\therefore$ singularity at $\cos z_0 = 0 \Rightarrow z_0 = \frac{(2k+1)}{2}\pi$, k integer

$\therefore z_0 = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ $\therefore |z_0| \geq \frac{\pi}{2} > 1$

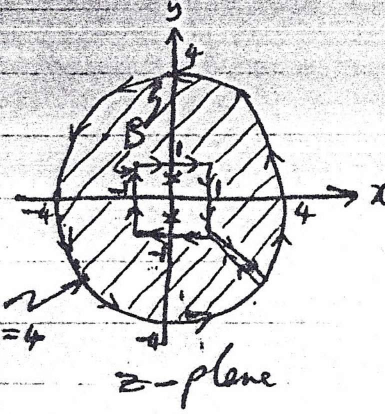
$\therefore$ C encloses no singularity of $\tan z$ $\therefore$ $\tan z$ is analytic within C $\therefore \oint_C \tan z\, dz = 0$.

29/127 (3) $f(z) = \dfrac{1}{3z^2+1}$ $\therefore f(z)$ is analytic in its domain of definition,

i.e., all z: $3z^2+1 \neq 0$

$\because 3z^2+1 = 0 \Rightarrow z^2 = -\frac{1}{3} = \frac{1}{3}\angle \pi + 2k\pi$

$\therefore z = \frac{1}{\sqrt{3}} \angle \frac{\pi}{2} + k\pi = \pm \frac{i}{\sqrt{3}} = \pm .58i$

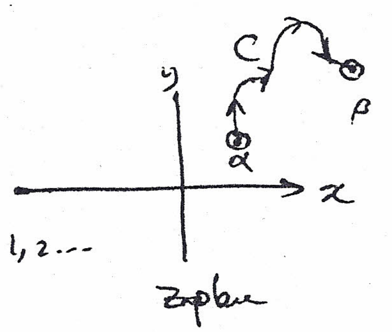


$\therefore f(z)$ is analytic throughout z : $z \neq \pm \frac{i}{\sqrt{3}}$

$\therefore f(z)$ is analytic on B and its interior

$\therefore \oint_B f(z)\,dz = 0$ $\qquad \therefore \oint_B \dfrac{dz}{3z^2+1} = 0$

5/127 (2) z^n is analytic throughout z-plane



$\therefore \displaystyle\int_{\substack{C \\ \alpha \to \beta}} z^n\,dz = \frac{z^{n+1}}{n+1}\Big|_\alpha^\beta = \frac{\beta^{n+1} - \alpha^{n+1}}{n+1}$, $n = 0, 1, 2 \cdots$

6/127 (a) (3) $e^{\pi z}$ is analytic everywhere,

$\therefore$ I) $\displaystyle\int_i^{i/2} e^{\pi z}\,dz = e^{\pi z}/\pi \Big|_i^{i/2} = \frac{1}{\pi}\left(e^{i\pi/2} - e^{i\pi}\right) = \frac{1}{\pi}(i+1) = (1+i)/\pi$

II) $\displaystyle\oint_{C:\,|z|=1} e^{\pi z}\,dz = 0$ because $e^{\pi z}$ is analytic within C.

(b) (2) $\cos\frac{z}{2}$ is analytic everywhere,

$\therefore \displaystyle\int_0^{\pi+2i} \cos\frac{z}{2}\,dz = 2\sin\frac{z}{2}\Big|_0^{\pi+2i} = 2\left(\sin\left(\frac{\pi}{2}+i\right) - \sin 0\right) = 2\left(\sin\frac{\pi}{2}\cosh 1 + i\underbrace{\cos\frac{\pi}{2}}_{0}\sinh 1\right)$

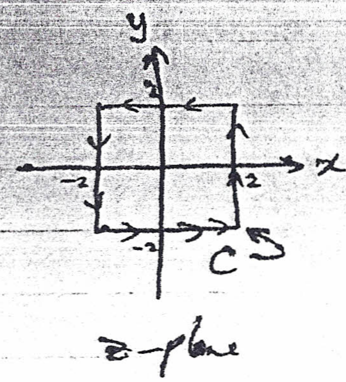
$= 2\left(\cosh 1 + i0 - 0\right) = 2\cosh 1 = 2 \cdot \frac{e^1 + e^{-1}}{2} = e + \frac{1}{e} = 3.08616$

(c) (2) $(z-2)^3$ is analytic everywhere,

$\therefore \displaystyle\int_1^3 (z-2)^3\,dz = \frac{(z-2)^4}{4}\Big]_1^3 = \frac{(3-2)^4 - (1-2)^4}{4} = \frac{1-1}{4} = 0$

3/138 (a)

$$\int_C \frac{e^{-z}}{z-\pi i/2}\,dz = \left(\text{Note: singularity point is at } z=\frac{\pi i}{2}\approx 1.6i \text{ and this is within } C\right)$$



$$= e^{-z}\Big|_{z=\frac{\pi i}{2}} * 2\pi i = e^{-\frac{\pi i}{2}}\cdot 2\pi i = -i * 2\pi i = 2\pi$$

(b)

$$\int_C \frac{\cos z\,dz}{z(z^2+8)} = \left(\text{singularity at } z=0,\ \pm 2\sqrt{2}\,i, \text{ and only } z=0 \text{ is within } C\right)$$

$$= \frac{\cos z}{z^2+8}\Big|_{z=0} * 2\pi i = \frac{\cos 0}{0+8} * 2\pi i = \frac{1}{8} * 2\pi i = \frac{\pi i}{4}$$

(c)

$$\int_C \frac{z\,dz}{2z+1} = \frac{z}{2}\Big|_{z=-\frac{1}{2}} * 2\pi i = -\frac{1}{4} * 2\pi i = -\pi i/2$$

(d)

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = \left[\tan(z/2)\right]'\Big|_{z=x_0} * \frac{2\pi i}{1} = \frac{1}{2}\sec^2(z/2)\Big|_{z=x_0} * 2\pi i = \pi i \sec^2(x_0/2)$$

$z=x_0$ within C

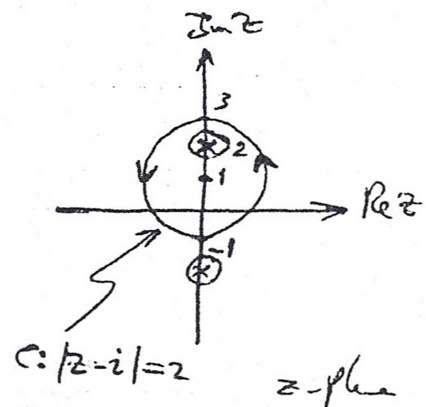
(e)

$$\int_C \frac{\cosh z\,dz}{z^4} = (\cosh z)'''\Big|_{z=0} * \frac{2\pi i}{3!} = (\sinh z)''\Big|_{z=0} * \frac{2\pi i}{6} = (\cosh z)'\Big|_{z=0} * \frac{\pi i}{3} =$$

$$= \sinh z\Big|_{z=0} * \frac{\pi i}{3} = \sinh 0 * \frac{i\pi}{3} = 0$$

4b/138

$$\oint_C g(z)\,dz = \oint_C \frac{dz}{(z^2+4)^2} =$$

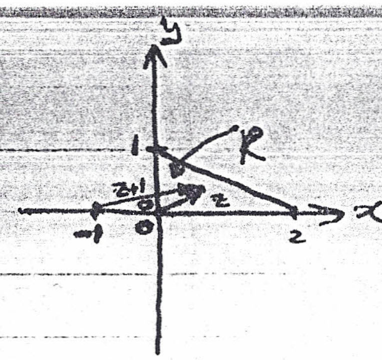


$$= \oint_C \frac{dz}{(z-2i)^2(z+2i)^2} = 2\pi i\left(\frac{1}{(z+2i)^2}\right)'\Bigg|_{2i} =$$

$$= 2\pi i\cdot\frac{-2}{(z+2i)^3}\Bigg|_{2i} = 2\pi i\cdot\frac{-2}{(4i)^3} = \frac{-4\pi i}{-64i} = \pi/16$$

$$\therefore \oint_C g(z)\,dz = \pi/16$$

10/139 (6) $f(z) = (z+1)^2 \quad \therefore |f(z)| = |z+1|^2$



$\therefore$ $|f(z)|$ is max. when $|z+1|$ is max.

but $|z+1|$ is max at $z=2$

$\therefore$ $|f(z)|$ is max. at $z=2$ where $|f(z)| = 9$

(Note: This is in agreement with max. value principle, because $|f(z)|$ is max. nowhere in the interior but on the boundary) z plane

Similarly, $|f(z)|$ is min. when $|z+1|$ is min., i.e at $z=0$

$\therefore$ $|f(z)|$ is min. at $z=0$ where $|f(z)| = 1$, $\therefore$ $1 \le \underset{z \in R}{|f(z)|} \le 9$

17/139 (8) $f(z)$ is entire, and $|f(z)| \le A|z| = M$ for all z: $|z| < \infty$

$\therefore$ According to Cauchy inequality; if C is the contour $|z - z_0| = r_0$,

then, $\left|\frac{2\pi i}{2!} f''(z_0)\right| = \left|\oint_C \frac{f(z)\,dz}{(z-z_0)^3}\right| \le \frac{M}{r_0^3} * 2\pi r_0 = \frac{2\pi M}{r_0^2}$

taking the limit as r_0 goes to ∞

$\therefore$ $\left|\frac{2\pi i\, f''(z_0)}{2!}\right| \le \lim_{r_0 \to \infty} \frac{2\pi M}{r_0^2} = \lim_{r_0 \to \infty} \frac{2\pi A|z|}{r_0^2} = \lim_{r_0 \to \infty} \frac{2\pi A}{r_0^2}|z - z_0 + z_0|$

$\le \lim_{r_0 \to \infty} \frac{2\pi A}{r_0^2}\left(|z - z_0| + |z_0|\right) = \lim_{r_0 \to \infty} \frac{2\pi A}{r_0^2}\left(r_0 + |z_0|\right) =$

$= 2\pi A \lim_{r_0 \to \infty} \left(\frac{1}{r_0} + \frac{|z_0|}{r_0^2}\right) = 0 \Rightarrow \quad \therefore \left|\frac{2\pi i}{2!} f''(z_0)\right| \le 0$

$\therefore$ $f''(z_0) = 0$ for any z_0 $\quad \therefore f(z) = az + b$

$\therefore$ $|f(z)| \le A|z| \Rightarrow |az + b| \le A|z|$ at $z=0 \Rightarrow |b| \le 0$

$\therefore$ $b = 0 \quad \therefore f(z) = az$

$\therefore$ $|az| = |a||z| \le A|z| \quad \therefore |a| \le A$

$\therefore$ for $|f(z)|$ to be $\le A|z|$, $f(z)$ must be $= az$

with $|a| \le A$

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19/140

7) (a) Let $f(z)=z^n-z_0^n \quad \therefore f(z_0)=z_0^n-z_0^n=0$

$\therefore (z-z_0)$ is a factor in $f(z)$ & hence $f(z)$ is divisible by $(z-z_0)$

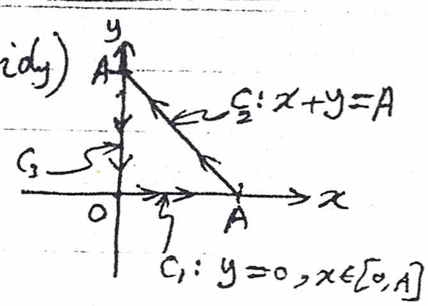
(b) Let $f(z)=P(z)-P(z_0) \quad \therefore f(z_0)=P(z_0)-P(z_0)=0$

$\therefore (z-z_0)$ is a factor in $f(z)$ & hence $f(z)$ is divisible by $(z-z_0)$ and the quotient is one degree less than $P(z)$

(c) If $P(z_0)=0$, then $f(z)$ (in (b)) is actually $P(z)$, $\therefore P(z)$ is divisible by $(z-z_0)$

#

$$\oint_C (z+\bar{z})\,dz=\oint_C[(x+iy)+(x-iy)](dx+idy)$$



$$=\oint_C 2x(dx+idy)=\int_{C_1+C_2+C_3} 2x(dx+idy)=$$

$$=\int_{C_1:\,y=0,\,x\in[0,A],\,dy=0} 2x\,(dx+idy)+\int_{C_2:\,x+y=A\,\Rightarrow\,dx+dy=0,\ x\in[A,0]} 2x\,(dx+idy)+\int_{C_3:\,x=0,\,dx=0\ \&\ y\in[A,0]} 2x(dx+idy)=$$

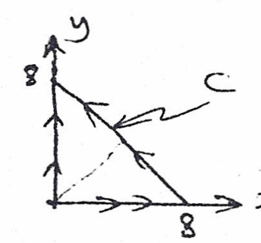
$$=\int_0^A 2x\,dx+\int_A^0 2x\,(dx-idx)+\int_A^0 0\,dy=$$

$$=x^2\Big|_0^A+(1-i)x^2\Big|_A^0+0=(A^2-0)+(1-i)(0-A^2)=A^2-A^2+iA^2=iA^2$$

$$\therefore \oint_C(z+\bar{z})\,dz=iA^2 \quad \left(\text{for } A=8 \ \therefore \oint_C=64i\right).$$

#

$$I=\oint_C\frac{dz}{z^2-8z+25}=\oint\frac{dz}{(z-4+3i)(z-4-3i)}$$



$$\left(z^2-8z+25=0\Rightarrow z=\frac{8\pm\sqrt{64-100}}{2}=4\pm i3\right)$$

$\therefore z=4-3i$ is in the fourth quadrant $\therefore$ Out of C.

$\therefore z=4+3i=5\angle 37$ in the first quadrant

& since all $\frac{1}{4}$ circles with radius $\le\frac{8}{\sqrt{2}}=5.66$ lie within C

$\therefore z=5\angle 37^\circ$ is within $C\Rightarrow z=4+3i$ is a singularity point.

$\therefore$ Applying Cauchy-theorem

$$\therefore I=\oint_C\frac{[1/(z-4+3i)]}{(z-4-3i)}\,dz=2\pi i\left[\frac{1}{z-4+3i}\right]_{z=4+3i}=\frac{2\pi i}{4+3i-4+3i}=\frac{2\pi i}{6i}=\frac{\pi}{3}$$

$$\therefore \oint\frac{dz}{z^2-8z+25}=\pi/3.$$

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اثبات نظريتي تيلور ولورنت

1. *Proof of Taylor Theorem:*

given: $f(z)$ is analytic in the domain $|z-z_0| \le r$ (circle, centre z_0, radius r)

show that $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$
in $|z-z_0| < r$

Proof:

Consider the sum: $S_n = 1 + w + w^2 + \cdots + w^n$,
It was shown in $\boxed{\frac{4}{145}}$ that $S_n = \frac{1-w^{n+1}}{1-w} = \frac{1}{1-w} - \frac{w^{n+1}}{1-w}$

$$\therefore \frac{1}{1-w} = S_n + \frac{w^{n+1}}{1-w} = 1 + w + w^2 + \cdots + w^n + \frac{w^{n+1}}{1-w}$$

Similarly:

$$\frac{1}{s-z} = \frac{1}{s-z+z_0-z_0} = \frac{1}{s-z_0-(z-z_0)} = \frac{1}{(s-z_0)\left(1-\frac{z-z_0}{s-z_0}\right)} =$$

$$= \frac{1}{s-z_0}\cdot\left(\frac{1}{1-\frac{z-z_0}{s-z_0}}\right) = \frac{1}{s-z_0}\cdot\left(1 + \frac{z-z_0}{s-z_0} + \left(\frac{z-z_0}{s-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{s-z_0}\right)^n + \frac{\left(\frac{z-z_0}{s-z_0}\right)^{n+1}}{1-\frac{z-z_0}{s-z_0}}\right)$$

$$= \frac{1}{s-z_0} + \frac{z-z_0}{(s-z_0)^2} + \frac{(z-z_0)^2}{(s-z_0)^3} + \cdots + \frac{(z-z_0)^n}{(s-z_0)^{n+1}} + \frac{\left(\frac{z-z_0}{s-z_0}\right)^{n+1}}{s-z_0-(z-z_0)}$$

multiplying by $f(s)$ both sides:

$$\therefore \frac{f(s)}{s-z} = \sum_{m=0}^{n} \frac{f(s)\cdot(z-z_0)^m}{(s-z_0)^{m+1}} + \frac{f(s)}{s-z}\cdot\left(\frac{z-z_0}{s-z_0}\right)^{n+1}$$

multiplying by ds and integrating around $C: |s-z_0| = r$

$$\therefore \oint_C \frac{f(s)\,ds}{s-z} = \oint_C \sum_{m=0}^{n} \frac{f(s)(z-z_0)^m\,ds}{(s-z_0)^{m+1}} + \oint_C \frac{f(s)}{(s-z)}\cdot\left(\frac{z-z_0}{s-z_0}\right)^{n+1} ds$$

$$= \sum_{m=0}^{n} (z-z_0)^m \oint_C \frac{f(s)\,ds}{(s-z_0)^{m+1}} + (z-z_0)^{n+1}\cdot\oint_C \frac{f(s)\,ds}{(s-z)(s-z_0)^{n+1}} \qquad (*)$$

but, according to Cauchy Theorem:

$$\oint_C \frac{f(s)\,ds}{(s-z)^{m+1}} = 2\pi i \frac{f^{(m)}(z)}{m!}$$

$$\therefore \oint_C \frac{f(s)ds}{s-z} = 2\pi i \frac{f^{(0)}(z)}{0!} = 2\pi i f(z) = \sum_{m=0}^{n} (z-z_0)^m \oint_C \frac{f(s)ds}{(s-z_0)^{m+1}} +$$

$$+ (z-z_0)^{n+1} \oint_C \frac{f(s)ds}{(s-z)(s-z_0)^{n+1}} = \sum_{m=0}^{n} (z-z_0)^m * 2\pi i \frac{f^{(m)}(z_0)}{m!} + R_n$$

where $R_n = (z-z_0)^{n+1} \oint_C \frac{f(s)ds}{(s-z)(s-z_0)^{n+1}}$

$$\therefore 2\pi i f(z) = 2\pi i \sum_{m=0}^{n} \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + R_n$$

$$\therefore f(z) = \sum_{m=0}^{n} \frac{f^{(m)}(z_0)}{m!}\cdot(z-z_0)^m + \frac{R_n}{2\pi i}$$

$|s-z_0| = r$
$|z-z_0| = r'$

Now:

$$|R_n| = \left| (z-z_0)^{n+1} \oint \frac{f(s)ds}{(s-z)(s-z_0)^{n+1}} \right| =$$

$$= (r')^{n+1} \oint_C \left| \frac{f(s)\,ds}{(s-z_0)-(z-z_0)} \right| \frac{1}{r^{n+1}} = \left(\frac{r'}{r}\right)^{n+1} \cdot \oint_C \frac{|f(s)\,ds|}{|(s-z_0)-(z-z_0)|}$$

$$\because |(s-z_0)-(z-z_0)| \geq \big||s-z_0|-|z-z_0|\big| = |r-r'| = r-r'$$

$$\therefore |R_n| = \left(\frac{r'}{r}\right)^{n+1} \oint_C \frac{|f(s)ds|}{|s-z_0-(z-z_0)|} \leq \left(\frac{r'}{r}\right)^{n+1} * \frac{M\,2\pi r}{r-r'},$$

where M is the maximum of $|f(s)|$ on C.

$$\therefore \left|\frac{R_n}{2\pi i}\right| = \left|\frac{R_n}{2\pi}\right| \leq M\left(\frac{r'}{r}\right)^{n+1} * \frac{r}{r-r'} \qquad (**)$$

$$\therefore \left|\frac{R_n}{2\pi i}\right| \to 0 \text{ as } n \to \infty \text{ provided } r' < r \quad \left(\frac{r'}{r} < 1\right)$$

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$$\therefore f(z) = \lim_{n\to\infty} \sum_{m=0}^{n} \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{R_n}{2\pi i} = \sum_{m=0}^{\infty} f^{(m)}(z_0)\cdot\frac{(z-z_0)^m}{m!} + \overset{0}{\frac{R_n}{2\pi i}}$$

$\therefore$ OK

2. Proof of Laurent Theorem:

given: $f(z)$ is analytic in the domain $r_1 \le |z - z_0| \le r_2$ (annulus, centre z_0, radii r_1 & r_2)

show that $f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$ where $C_n = \frac{1}{2\pi i} \oint_C \frac{f(s)\, ds}{(s - z_0)^{n+1}}$

$C: r_1 < |z - z_0| = r' < r_2$

Let:

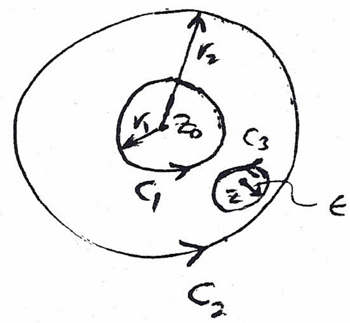
C_1 be $|s - z_0| = r_1$

C_2 be $|s - z_0| = r_2$

C_3 be $|s - z| = \epsilon$: $r_1 < |z - z_0| < r_2$

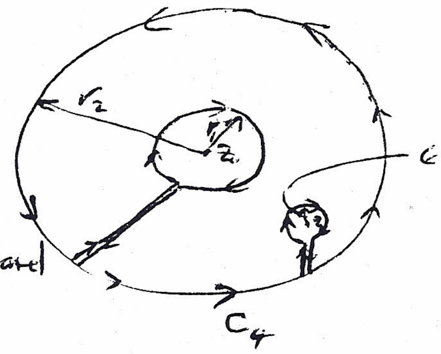
C_4 be the contour shown in the figure

where ϵ is a very small positive number.



$\therefore \oint_{C_4} \frac{f(s)\, ds}{s - z} = 0$ because $\frac{f(s)}{s - z}$ is

analytic throughout the region bounded by C_4



$\because \oint_{C_4} = \oint_{C_2 - C_1 - C_3} = \oint_{C_2} - \oint_{C_1} - \oint_{C_3}$ of the above integrand

$\therefore \oint_{C_2} - \oint_{C_1} - \oint_{C_3} = 0 \Rightarrow \therefore \oint_{C_3} = \oint_{C_2} - \oint_{C_1}$ of the above integrand

$$\therefore \oint_{C_3} \frac{f(s)\, ds}{s - z} = \oint_{C_2} \frac{f(s)\, ds}{s - z} - \oint_{C_1} \frac{f(s)\, ds}{s - z}$$

but according to Cauchy theorem, $\because f(s)$ is analytic within C_3 and on it,

$$\therefore \oint_{C_3} \frac{f(s)\, ds}{s - z} = 2\pi i * f(z)$$

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$$\therefore \oint_{C_3} \frac{f(s)\, ds}{s - z} = 2\pi i f(z) = \oint_{C_2} \frac{f(s)\, ds}{s - z} - \oint_{C_1} \frac{f(s)\, ds}{s - z}$$

$$\therefore f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(s)\, ds}{s - z} + \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)\, ds}{z - s} = \text{(as in Taylor proof*)}$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{n}(z-z_0)^m \oint_{C_2}\frac{f(s)\,ds}{(s-z_0)^{m+1}} + \frac{1}{2\pi i}(z-z_0)^{n+1}\cdot\oint_{C_2}\frac{f(s)\,ds}{(s-z)(s-z_0)^{n+1}}$$

$$+\frac{1}{2\pi i}\sum_{m=0}^{n}\frac{1}{(z-z_0)^{m+1}}\oint_{C_1} f(s)(s-z_0)^m\,ds + \frac{1}{2\pi i}\cdot\frac{1}{(z-z_0)^{n+1}}\cdot\oint_{C_1}\frac{(s-z_0)^{n+1}}{(z-s)}f(s)\,ds$$

(Note: The second part is obtained by interchanging s & z in $(*)$)

$$= \frac{1}{2\pi i}\sum_{m=0}^{n}(z-z_0)^m \oint_{C_2}\frac{f(s)\,ds}{(s-z_0)^{m+1}} + R_{2_n}$$

$$+\frac{1}{2\pi i}\sum_{m=0}^{n}\frac{1}{(z-z_0)^{m+1}}\oint_{C_1} f(s)\cdot(s-z_0)^m\,ds + R_{1_n}$$

where $R_{2_n} = \frac{1}{2\pi i}(z-z_0)^{n+1}\oint_{C_2}\frac{f(s)\,ds}{(s-z)(s-z_0)^{n+1}}$

& $R_{1_n} = \frac{1}{2\pi i}\cdot\frac{1}{(z-z_0)^{n+1}}\cdot\oint_{C_1}\frac{(s-z_0)^{n+1}}{(z-s)}f(s)\,ds$

Now: assuming $|z-z_0| = r'$ (Note $r_1 < r' < r_2$)

$\therefore$ $|R_{2_n}| =$ (as in $(**)$)

$$\leq M_2\left(\frac{r'}{r_2}\right)^{n+1} * \frac{r_2}{r_2 - r'}$$, where M_2 is maximum $|f(s)|$ on C_2

$\because$ $r' < r_2$ $\therefore$ $\frac{r'}{r_2} < 1$ & R_{2_n} converges to zero as $n \to \infty$

& $|R_{1_n}| \leq M_1\left(\frac{r_1}{r'}\right)^{n+1}\frac{r_1}{r'-r_1}$, where M_1 is maximum $|f(s)|$ on C_1

$\because$ $r_1 < r'$ $\therefore$ $\frac{r_1}{r'} < 1$ & R_{1n} converges to zero as $n \to \infty$

$$\therefore f(z) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}(z-z_0)^m\oint_{C_2}\frac{f(s)\,ds}{(s-z_0)^{m+1}} + \cancel{R_{2_\infty}}^{\,0}$$

$$+\frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{1}{(z-z_0)^{m+1}}\oint_{C_1}(s-z_0)^m f(s)\,ds + \cancel{R_{1_\infty}}^{\,0}$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty}(z-z_0)^m\oint_{C_2}\frac{f(s)\,ds}{(s-z_0)^{m+1}} + \frac{1}{2\pi i}\sum_{m=0}^{\infty}(z-z_0)^{-m-1}\oint_{C_1}\frac{f(s)\,ds}{(s-z_0)^{-m}}$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty}(z-z_0)^m\oint_{C_2}\frac{f(s)\,ds}{(s-z_0)^{m+1}} + \frac{1}{2\pi i}\sum_{-m=0}^{-\infty}(z-z_0)^{-m-1}\oint_{C_1}\frac{f(s)\,ds}{(s-z_0)^{-m}} =$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty}(z-z_0)^m \oint_{C_2}\frac{f(s)ds}{(s-z_0)^{m+1}} + \frac{1}{2\pi i}\sum_{k=0}^{-\infty}(z-z_0)^{k-1}\oint_{C_1}\frac{f(s)ds}{(s-z_0)^{k}} =$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty}(z-z_0)^m \oint_{C_2}\frac{f(s)ds}{(s-z_0)^{m+1}} + \frac{1}{2\pi i}\sum_{k-1=-1}^{-\infty}(z-z_0)^{k-1}\oint_{C_1}\frac{f(s)ds}{(s-z_0)^{k-1+1}} =$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty}(z-z_0)^m \oint_{C_2}\frac{f(s)ds}{(s-z_0)^{m+1}} + \frac{1}{2\pi i}\sum_{n=-1}^{-\infty}(z-z_0)^n\oint_{C_1}\frac{f(s)ds}{(s-z_0)^{n+1}} =$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_2}\frac{f(s)ds}{(s-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}(z-z_0)^n\oint_{C_1}\frac{f(s)ds}{(s-z_0)^{n+1}}$$

But: $\displaystyle\oint_{C_5}\frac{f(s)\,ds}{(s-z_0)^{n+1}} = 0$ because

$\displaystyle\frac{f(s)}{(s-z_0)^{n+1}}$ is analytic within and on C_5

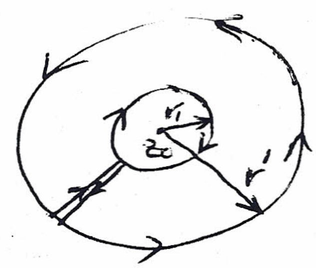
C_5 $r_1 < r' < r_2$

$$\therefore \oint_{C_5} = \oint_{C_2 - C} = \oint_{C_2} - \oint_{C} \quad \text{where } C: |z-z_0| = r'$$

$$\therefore \oint_{C_5} = 0 = \oint_{C_2} - \oint_{C} \Rightarrow \oint_{C_2} = \oint_{C}$$

$$\therefore \oint_{C_2}\frac{f(s)ds}{(s-z_0)^{n+1}} = \oint_{C}\frac{f(s)ds}{(s-z_0)^{n+1}}$$

Similarly: $\displaystyle\oint_{C_6}\frac{f(s)ds}{(s-z_0)^{n+1}} = 0$



$$\therefore \oint_{C_6} = \oint_{C-C_1} = \oint_{C} - \oint_{C_1} \quad \therefore \oint_{C} - \oint_{C_1} = 0$$

C_6 $r_1 < r' < r_2$

$$\therefore \oint_{C_1}\frac{f(s)\,ds}{(s-z_0)^{n+1}} = \oint_{C}\frac{f(s)ds}{(s-z_0)^{n+1}}$$

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$$\therefore f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C}\frac{f(s)ds}{(s-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}(z-z_0)^n\oint_{C}\frac{f(s)ds}{(s-z_0)^{n+1}}$$

$$= \sum_{n=-\infty}^{\infty}(z-z_0)^n\underbrace{\left[\frac{1}{2\pi i}\oint_{C}\frac{f(s)ds}{(s-z_0)^{n+1}}\right]}_{C_n} = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n$$

$\therefore$ OK

(Note: Taylor Series is a special case of Laurent Series when $f(s)$ is analytic within ...)

$\boxed{\frac{1}{144}}$ $z_n = -2 + i\frac{(-1)^n}{n^2} = x_n + iy_n$

$\therefore$ $x_n = -2$ (constant converges to itself.)

& $y_n = \frac{(-1)^n}{n^2}$ $\therefore$ $\lim_{n\to\infty} y_n = \frac{(-1)^n}{\infty} = \pm 0$

$\therefore$ y_n converges to zero

$\therefore$ x_n & y_n converges to -2 & 0 respectively,

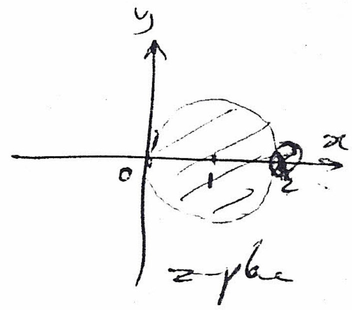
$\therefore$ z_n converges to $-2 + i0 = -2$

OR # You can prove it by proving $|z_n - z| < \epsilon$ for $n > N$

$\therefore \left|\left(-2 + i\frac{(-1)^n}{n^2}\right) - (-2)\right| < \epsilon \Rightarrow \frac{1}{n^2} < \epsilon$ $\therefore N = \frac{1}{\sqrt{\epsilon}}$

$\therefore$ Choosing $n > \frac{1}{\sqrt{\epsilon}}$ gives $|z_n - z| < \epsilon$ $\therefore$ z_n converges to z.

$\boxed{\#}$ $f(z) = \frac{1}{z-2}$, $|z-1| < 1$



$\therefore$ Singularity at $z=2$ is out of region.

$\therefore$ $f(z)$ is analytic within the region, $|z-1|<1$

$\therefore$ Taylor series with $z_0 = 1$.

$\therefore$ $f(z_0) = -1$ & $f'(z_0) = -\frac{1}{(z_0-2)^2} = -1$ & $f''(z_0) = \frac{2}{(z_0-2)^3} = -2$

& $f^{(n)}(z_0) = \frac{(-1)^n n!}{(z_0-2)^{n+1}} = -n!$

$\therefore$ $\frac{1}{z-2} = -1 - \frac{z-1}{1!} - 2\frac{(z-1)^2}{2!} + \cdots - \frac{n!\,(z-1)^n}{n!} + \cdots$

$= -\sum_{k=0}^{\infty} (z-1)^k$ for convergence $|z-1| < 1$ (OK)

OR: $\frac{1}{z-2} = \frac{1}{z-1-1} = -\frac{1}{1-(z-1)} = -\left[1 + (z-1) + (z-1)^2 + \cdots + (z-1)^n \cdots\right]$

$\boxed{\frac{4}{145}}$ Let $1+z+z^2+\dots\dots+z^n = S_n$ ①

$\therefore z+z^2+z^3+\dots\dots+z^{n+1} = z\,S_n$ ②

$\therefore$ ① $-$ ② $\Rightarrow 1-z^{n+1} = S_n(1-z)$

$\therefore S_n = \dfrac{1-z^{n+1}}{1-z}$ $\quad\therefore \lim_{n\to\infty} S_n = \dfrac{1}{1-z}$ provided $|z|<1$

$\therefore z+z^2+z^3+\dots\dots+z^{n+1}+\dots\dots z^{\infty} =$ (as in ②) $z\cdot S_\infty = \dfrac{z}{1-z}$

$\therefore \displaystyle\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$ $\quad(|z|<1)$

Putting $z = r\angle\theta = r(\cos\theta + i\sin\theta)$ $\therefore z^n = r^n\angle n\theta = r^n(\cos n\theta + i\sin n\theta)$

$\therefore \displaystyle\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty} r^n(\cos n\theta + i\sin n\theta) =$ (provided $|z|=r<1$)

$$= \frac{z}{1-z} = \frac{r(\cos\theta+i\sin\theta)}{1-r(\cos\theta+i\sin\theta)} = r.\frac{(\cos\theta+i\sin\theta)}{(1-r\cos\theta)-ir\sin\theta}\cdot\frac{1-r\cos\theta+ir\sin\theta}{1-r\cos\theta+ir\sin\theta} =$$

$$= r.\frac{\cos\theta(1-r\cos\theta)-r\sin^2\theta + i(\sin\theta(1-r\cos\theta)+r\sin\theta\cos\theta)}{(1-r\cos\theta)^2+(r\sin\theta)^2} =$$

$$= r.\frac{\cos\theta - r(\cos^2\theta+\sin^2\theta) + i(\sin\theta - r\sin\theta\cos\theta + r\sin\theta\cos\theta)}{1-2r\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta} =$$

$$= r.\frac{\cos\theta - r + i\sin\theta}{1-2r\cos\theta+r^2} = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} + i\,\frac{r\sin\theta}{1-2r\cos\theta+r^2}.$$

Equating corresponding components of the series & sum,

$$\therefore \sum_{n=1}^{\infty} r^n\cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}. \quad \text{(real)}$$

$$\& \sum_{n=1}^{\infty} r^n\sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2}. \quad \text{(imaginary)}$$

$\Big\}\ 0<r<1$

$\frac{2a}{149}$ $\quad \frac{1}{z^2} = f(z)$, Taylor at $z_0 = -1$

$$\therefore f(z) = \frac{1}{z^2} = f(-1) + f'(-1)(z+1) + f''(-1)(z+1)^2/2 + \cdots + f^{(n)}(-1)(z+1)^n/n!$$

$$= 1 - 2z^{-3}\Big|_{-1}\cdot(z+1) + 6z^{-4}\Big|_{-1}\frac{(z+1)^2}{2} + \cdots + (-1)^n(n+1)!\,z^{-(n+2)}\Big|_{-1}\frac{(z+1)^n}{n!} + \cdots$$

$$= 1 + 2(z+1) + 3(z+1)^2 + \cdots + (-1)^n(n+1)(-1)^{-n-2}\cdot(z+1)^n + \cdots$$

$$= 1 + 2(z+1) + 3(z+1)^2 + \cdots + (n+1)(z+1)^n + \cdots$$

$$= \sum_{n=0}^{\infty}(n+1)(z+1)^n \qquad \text{convergent when } |z+1| < 1$$

$\frac{2b}{149}$ $\quad \because$ RHS involves $(z-2)^n$ $\therefore z_0 = 2$

$\therefore$ Expanding $\frac{1}{z^2}$ by Taylor series about $z_0 = 2$ in the domain $|z-2| < 2$ where $\frac{1}{z^2}$ is analytic (because, $z=0$ is not in $|z-2|<2$).

$$\therefore \frac{1}{z^2} = \frac{1}{z^2}\Big|_{z=2} + \left(-2z^{-3}\Big|_{z=2}\right)\frac{(z-2)}{1!} + \left(6z^{-4}\Big|_{z=2}\right)\frac{(z-2)^2}{2} + \cdots + \left((-1)^n\cdot z^{-n-2}\cdot(n+1)!\Big|_{z=2}\right)\frac{(z-2)^n}{n!}$$

$$= \frac{1}{4} - \frac{1}{4}(z-2) + \frac{3}{16}(z-2)^2 + \cdots + (-1)^n\cdot(2)^{-n-2}\cdot(n+1)\cancel{n!}\cdot\frac{(z-2)^n}{\cancel{n!}} + \cdots$$

$$= \frac{1}{4} - \frac{z-2}{4} + \frac{3}{16}(z-2)^2 + \cdots + (-1)^n\frac{(n+1)(z-2)^n}{2^n\cdot 2^2} + \cdots$$

$$= \frac{1}{4} + (-1)^1\frac{(z-2)^1 2}{4*2^1} + \frac{(-1)^2(z-2)^2*3}{4*2^2} + \cdots + (-1)^n\frac{(z-2)^n\cdot(n+1)}{4*(2^n)} + \cdots \text{ n integer}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(n+1)\cdot\left(\frac{2-z}{2}\right)^n, \quad \therefore \text{OK}$$

$\boxed{\frac{4}{149}}$ $\sinh z = f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0) \cdot \frac{(z-z_0)^n}{n!}$, $z_0 = \pi i$

$f(z_0)\Big|_{\pi i} = \sinh \pi i = i \sin \pi = 0$

$f' = (\sinh z)'\Big|_{\pi i} = \cosh z\Big|_{\pi i} = \cosh \pi i = \cos \pi = -1$

$f'' = \sinh z\Big|_{\pi i} = \sinh i\pi = i \sin \pi = 0$

$f''' = \cosh \pi i = \cos \pi = -1$

$\therefore f^{(2k+1)}(\pi i) = -1$, $f^{(2k)}(\pi i) = 0$

$$= \Big(\sum_{\substack{n=0 \\ \text{even}}}^{\infty} + \sum_{\substack{n=1 \\ \text{odd}}}^{\infty}\Big) f^{(n)}(z_0) \cdot \frac{(z-\pi i)^n}{n!} =$$

$$= \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} -1 \cdot \frac{(z-\pi i)^n}{n!} = -\sum_{\substack{2k+1=1 \\ \text{odd}}}^{\infty} \frac{(z-\pi i)^{2k+1}}{(2k+1)!}$$

$$= -\sum_{k=0}^{\infty} \frac{(z-\pi i)^{2k+1}}{(2k+1)!}$$

$\boxed{\frac{6}{149}}$ $\frac{1}{4z - z^2} = \frac{1}{z(4-z)} = \frac{A}{z} + \frac{B}{4-z}$

$\therefore A(4-z) + Bz = 1$ $\therefore$ at $z=0$ $4A = 1$ $\therefore A = 1/4$

& at $z = 4$ $\therefore$ $4B = 1$ $\therefore B = 1/4$

$$\therefore \frac{1}{4z - z^2} = \frac{1}{4}\Big[\frac{1}{z} + \frac{1}{4-z}\Big] = \frac{1}{4z} + \frac{1}{16(1-\frac{z}{4})} =$$

$$= \frac{1}{4z} + \frac{1}{16}\Big[1 + \frac{z}{4} + \cdots + \Big(\frac{z}{4}\Big)^n\Big]$$

$z \neq 0$ & $\left|\frac{z}{4}\right| < 1$ i.e $0 < |z| < 4$

$$= \frac{1}{4z} + \frac{1}{16} \cdot \sum_{n=0}^{\infty} \Big(\frac{z}{4}\Big)^n = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}} =$$

$$= \frac{1}{4z} + \sum_{n+1=1}^{\infty} \frac{z^{n+1-1}}{4^{n+1+1}} = \frac{1}{4z} + \sum_{k=1}^{\infty} \frac{z^{k-1}}{4^{k+1}} = \sum_{k=0}^{\infty} \frac{z^{k-1}}{4^{k+1}}$$

$$\therefore \frac{1}{4z - z^2} = \sum_{k=0}^{\infty} \frac{z^{k-1}}{4^{k+1}} \quad \text{where } 0 < |z| < 4$$

\# $f(z) = \frac{1}{z-1}$, $|z-2| < 1$

$$\therefore f(z) = \frac{1}{z-2+1} = \frac{1}{1+(z-2)} = 1 - (z-2) + (z-2)^2 - (z-2)^3 \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k (z-2)^k, \quad |z-2| < 1 \text{ for convergence.}$$

$\frac{9}{150}$ Consider first $\sin w$ expanded by Maclaurin series:

$$\therefore \sin w = (\sin w)\Big|_0 + (\cos w)\Big|_0 \frac{w}{1!} + (-\sin w)\Big|_0 \frac{w^2}{2!} + \cdots + \left[(\sin w)^{(n)}\Big|_0\right] \cdot \frac{w^n}{n!} + \cdots$$

$$= +0 + \frac{w}{1!} - 0 - \frac{w^3}{3!} + 0 + \frac{w^5}{5!} - 0 - \frac{w^7}{7!} + \cdots \frac{(-1)^{n-1} w^{2n-1}}{(2n-1)!} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} * \frac{w^{2n-1}}{(2n-1)!}$$

$\therefore$ putting $w = z^2$ $\therefore$ $\sin(z^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z^2)^{2n-1}}{(2n-1)!} = \sum_{n=1}^{\infty} (-1)^{n-1} * \frac{z^{4n-2}}{(2n-1)!}$

dividing by z^4 $\therefore$ $\frac{\sin(z^2)}{z^4} = \sum_{n=1}^{\infty} (-1)^{n-1} * \frac{z^{4n-6}}{(2n-1)!}$, n integer.

$$\therefore \frac{\sin(z^2)}{z^4} = z^{-2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \frac{z^{14}}{9!} - \frac{z^{18}}{11!} + \cdots (-1)^{n-1} \frac{z^{4n-6}}{(2n-1)!} + \cdots$$

$\therefore$ OK

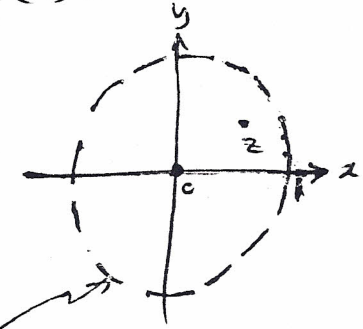
$\boxed{\frac{3}{161}}$ $\quad \frac{1}{1+s} = \frac{1}{1-(-s)} = 1+(-s)+(-s)^2+(-s)^3+ \cdots +(-s)^n+ \cdots$ $\quad |s|<1$

$= 1-s+s^2-s^3+ \cdots (-1)^n s^n + \cdots = \sum_{n=0}^{\infty}(-1)^n s^n$

$\therefore \int_0^z \frac{ds}{1+s} = \int_0^z (1-s+s^2-s^3+\cdots(-1)^n s^n+\cdots)\,ds =$

$= \left. s-\frac{s^2}{2}+\frac{s^3}{3}-\frac{s^4}{4}+\cdots (-1)^n\frac{s^{n+1}}{n+1}+\cdots \right]_0^z$

$= z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\cdots(-1)^n\frac{z^{n+1}}{n+1}+\cdots$



$|z|<1$ z-plane

$= \sum_{n=0}^{\infty}(-1)^n\frac{z^{n+1}}{n+1} = \sum_{n+1=1}^{\infty}(-1)^{n+1+1}\frac{z^{n+1}}{n+1} =$

$= \sum_{m=1}^{\infty}(-1)^{m+1}\frac{z^m}{m}$

but $\int_0^z \frac{ds}{1+s} = \log(1+s)\Big]_0^z$ in $|s|<1$ because $\frac{1}{1+s}$ is analytic within $|s|<1$

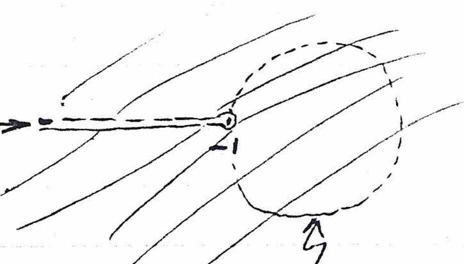
$\therefore \quad = \log(1+z) - \log 1$ provided $\log(1+z)$ is analytic within $|z|<1$

$\because$ Log $(1+z)$ is analytic within $|z|<1$

$\therefore \int_0^z \frac{ds}{1+s} = \text{Log}(1+z) - \cancel{\text{Log}\,1}^{\,0}$

$\therefore \text{Log}(1+z) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{z^m}{m}$ $\therefore$ OK

singularity of Log(z+1) is at $y=0$, $x\leq -1$



Log(z+1) is analytic within $|z|<1$

(check: expand Log $(1+z)$ by Maclaurin expansion

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$\therefore \text{Log}(1+z) = \left(\text{Log}(1+z)\right)\Big|_0 + \left(\frac{1}{1+z}\right)\Big|_0 z + \left(-1(1+z)^{-2}\right)\Big|_0\frac{z^2}{2!} + \left(-1\times(-2)(1+z)^{-3}\right)\Big|_0\frac{z^3}{3!}+\cdots$

$= 0+z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\cdots(-1)^{m+1}\frac{z^m}{m}+\cdots = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{z^m}{m}$)

$\therefore$ OK

$\boxed{\frac{8}{162}}$: $f(z)$ is analytic at z_0 $\therefore$ Expanding about z_0 by Taylor series

$$\therefore \quad f(z) = f(z_0) + f'(z_0)(z-z_0) + f''(z_0)\cdot\frac{(z-z_0)^2}{2} + \cdots + f^{(n)}(z_0)\frac{(z-z_0)^n}{n!} + \cdots$$

dividing by $z-z_0$

$$\therefore \quad \frac{f(z)}{(z-z_0)} = \frac{\cancel{f(z_0)}^{0}}{z-z_0} + f'(z_0) + f''(z_0)\frac{(z-z_0)}{2} + \cdots + f^{(n)}(z_0)\cdot\frac{(z-z_0)^{n-1}}{n!} + \cdots$$

$$= (\text{since } f(z_0)=0) \quad f'(z_0) + f''(z_0)\frac{z-z_0}{2} + \cdots + f^{(n)}(z_0)\cdot\frac{(z-z_0)^{n-1}}{n!} + \cdots$$

$$\therefore \quad \lim_{z\to z_0}\frac{f(z)}{z-z_0} = \lim_{z\to z_0} f'(z_0) + \lim_{z\to z_0} f''(z_0)\frac{z-z_0}{2} + \cdots + \lim_{z\to z_0} f^{(n)}(z_0)\frac{(z-z_0)^{n-1}}{n!} + \cdots$$

$$= f'(z_0) + f''(z_0)(0) + \cdots + f^{(n)}(z_0)\cdot(0) + \cdots$$

$$= f'(z_0)$$

$\therefore$ When $f(z)$ is analytic at z_0 & $f(z_0)=0$, then $\lim_{z\to z_0}\frac{f(z)}{z-z_0} = f'(z_0)$

(Note $f'(z_0) = \lim_{z\to z_0}\frac{f(z) - \cancel{f(z_0)}^{0}}{z-z_0}$, another proof).

$\boxed{\frac{4}{168}}$ (a) $f(z) = \frac{z+1-1+1}{z-1} = \frac{z-1}{z-1} + \frac{z}{z-1} = 1 - \frac{z}{1-z} = 1 - z\left(\frac{1}{1-z}\right) = 1 - z(1 + z + z^2 + \cdots + z^n + \cdots)$

$$= 1 - z\sum_{n=0}^{\infty} z^n = 1 - z - z\sum_{n=1}^{\infty} z^n = -1 - z\sum_{n=1}^{\infty} z^n$$

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$$\therefore \quad f(z) = -1 - z\sum_{n=1}^{\infty} z^n \qquad \text{provided } |z| < 1$$

(b) $f(z) = 1 - z\left(\frac{1}{1-z}\right)$, expanding $\frac{1}{1-z}$ by Laurent

Annulus domain

Series about $z_0=0$, in the annulus shown where it is analytic, $\therefore \frac{1}{1-z} = \sum_{-\infty}^{\infty} C_n z^n$

$$\therefore \quad C_n = \frac{1}{2\pi i}\oint_{C:|s-z_0|=r>1}\frac{f(s)\,ds}{(s-z_0)^{n+1}} = \frac{1}{2\pi i}\oint_C \frac{\frac{1}{1-s}}{s^{n+1}}\,ds = \frac{1}{2\pi i}\oint_C \frac{ds}{s^{n+1}(1-s)} = (\text{for } n \geq 0)$$

$$= \frac{1}{2\pi i} \oint_C \left(\frac{A}{1-s} + \frac{B_0 + B_1 s + B_2 s^2 + \cdots + B_n s^n}{s^{n+1}} \right) ds =$$

(Comparing coeffecients $\therefore A s^{n+1} + (1-s)(B_0 + \cdots + B_n s^n) = 1$

$\therefore B_0 = 1$ because $s^0 = s^0$

$-B_0 + B_1 = 0$ " $s^1 = s^1 \Rightarrow B_1 = B_0 = 1$

$-B_1 + B_2 = 0$ " $s^2 = s^2 \Rightarrow B_2 = B_1 = 1$

$\vdots$

$-B_j + B_{j+1} = 0$ " $s^{j+1} = s^{j+1} \Rightarrow B_{j+1} = B_j = 1$, $j < n$

$\vdots$

$-B_n + A = 0$ " $s^{n+1} = s^{n+1} \Rightarrow B_n = A \therefore A = B_n = 1$

$$\therefore C_n = \frac{1}{2\pi i} \oint_C \left(\frac{1}{1-s} + \frac{1+s+\cdots+s^n}{s^{n+1}} \right) ds$$

$$= \frac{1}{2\pi i} \oint_C \frac{-ds}{s-1} + \frac{1}{2\pi i} \sum_{k=0}^{n} \oint_C \frac{s^k}{s^{n+1}} ds = \frac{1}{2\pi i} (-1)\Big|_{s=1} * 2\pi i + \frac{1}{2\pi i} \sum_{k=0}^{n} 2\pi i \frac{(s^k)^{(n)}}{n!}\Bigg|_{s=0}$$

$$= -1 + \frac{1}{n!} \sum_{k=0}^{n} (s^k)^{(n)}\Big|_{s=0} = -1 + \frac{1}{n!} \left(0+0+0+0+\cdots+0+n!\right)\Big|_{s=0} =$$

$$= -1 + 1 = 0 \qquad (\text{for } n \geq 0)$$

$$\text{\&} \quad C_n = (\text{for } n < 0) \; \frac{1}{2\pi i} \oint_C \frac{s^{-n-1} ds}{1-s} = \frac{1}{2\pi i} \oint \frac{-s^{-n-1} ds}{s-1} = \frac{1}{2\pi i} * 2\pi i \left(-s^{-n-1}\right)\Big|_{s=1} =$$

$$= -1 \quad , \therefore \frac{1}{1-z} = \sum_{-\infty}^{\infty} C_n z^n = \sum_{n \geq 0} \cancel{C_n}^{\,0} z^n + \sum_{n<0} \cancel{C_n}^{\,-1} z^n$$

$$= -\sum_{n=-1}^{-\infty} z^n = -\sum_{-n=1}^{\infty} z^{-(-n)} = -\sum_{m=1}^{\infty} z^{-m}$$

$$\therefore f(z) = \frac{z+1}{z-1} = 1 - 2\left(\frac{1}{1-z}\right) = 1 - 2 * \left(-\sum_{m=1}^{\infty} z^{-m}\right) = 1 + 2\sum_{m=1}^{\infty} z^{-m}$$

$\therefore$ The Laurent Series expansion of $\frac{z+1}{z-1}$ (in $|z|>1$) $= 1 + 2\sum_{m=1}^{\infty} z^{-m}$

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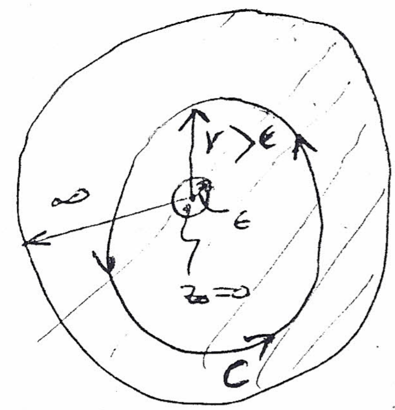
$\boxed{\frac{6}{168}}$

$$\frac{\sinh z}{z^2} = \sum_{-\infty}^{\infty} C_n z^n \qquad (\text{Note: } z_0 = 0)$$



Annulus domain

$$\therefore C_n = \frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{(s-z_0)^{n+1}} =$$

$$= \frac{1}{2\pi i} \oint_C \frac{\left(\frac{\sinh s}{s^2}\right) ds}{(s-0)^{n+1}} =$$

$$= \frac{1}{2\pi i} \oint_C \frac{\sinh s}{s^{n+3}} \cdot ds$$

$$= (\text{for } n \geq -2) \ \frac{1}{2\pi i} \left(\frac{(\sinh s)^{(n+2)}}{(n+2)!} \Big|_{s=0} \right) * 2\pi i \qquad (\text{Cauchy Theorem})$$

$$= \frac{1}{(n+2)!} * \begin{cases} 0 & n \text{ even} \\ 1 & \text{odd} \end{cases}$$

$$= (\text{for } n < -2) \ \frac{1}{2\pi i} \oint_C \left(s^{-n-3} \sinh s\right) ds = 0 \qquad (s^{-n-3} \sinh s \text{ is analytic within } C)$$

$$\therefore \frac{\sinh z}{z^2} = \sum_{-\infty}^{\infty} C_n z^n = \sum_{-\infty}^{-3} \cancel{C_n}^{\,0} z^n + \sum_{-2}^{\infty} C_n z^n =$$

$$= \sum_{\substack{n=-2 \\ n \text{ even}}}^{\infty} \cancel{C_n}^{\,0} z^n + \sum_{\substack{n=-1 \\ n \text{ odd}}}^{\infty} C_n z^n = \sum_{\substack{n=2k-1=-1 \\ n \text{ odd}}}^{\infty} C_{2k-1} z^{2k-1} =$$

$$= \sum_{\substack{k=0 \\ k \text{ integer}}}^{\infty} C_{2k-1} z^{2k-1} = \sum_{k=0}^{\infty} \frac{1}{(2k-1+2)!} z^{2k-1} = \sum_{k=0}^{\infty} \frac{z^{2k-1}}{(2k+1)!}$$

$$= z^{-1} + \frac{z}{3!} + \frac{z^3}{5!} + \cdots + \frac{z^{2k-1}}{(2k+1)!} + \cdots, \ \therefore \text{OK}$$

(Note: you can solve also by Taylor $\sinh z$

$$\therefore \sinh z = \left(\sinh z\Big|_0\right) + \left((\sinh z)'\Big|_0\right) * z + \cdots + \left((\sinh z)^{(n)}\Big|_0\right) \frac{z^n}{n!} + \cdots$$

$$= 0 + z + 0 + \frac{z^3}{3!} + \cdots + 0 \cdot \frac{z^{2k}}{(2k)!} + 1 \cdot \frac{z^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

(in the circular domain $|z| < \infty$

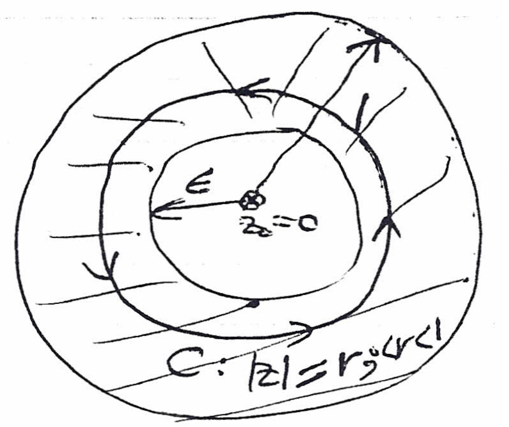
$$\therefore \frac{\sinh z}{z^2} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{z^{2k-1}}{(2k+1)!}$$

This is valid at $(|z| < \infty) \cap (z \neq 0) = 0 < |z| < \infty$, $\therefore$ OK.

$\boxed{\frac{9}{168}}$

$$\frac{e^z}{z(z^2+1)} = \sum_{n=-\infty}^{\infty} C_n (z-0)^n$$

$$= \sum_{n=-\infty}^{\infty} C_n z^n$$



$$\therefore \quad C_n = \frac{1}{2\pi i}\oint_C \frac{e^s/(s(s^2+1))}{(s-0)^{n+1}}\, ds$$

$$= \frac{1}{2\pi i}\oint_C \frac{e^s/(s^2+1)}{s^{n+2}}\, ds =$$

Annulus $0<|z|<1$

$= \left(\text{when } n<-1\right) = 0$ (because the integrand is analytic within C)

$$= \left(\text{when } n\geq -1\right) \frac{1}{\cancel{2\pi i}} * \cancel{2\pi i} * \frac{\left[e^s/(s^2+1)\right]^{(n+1)}}{(n+1)!}\Bigg|_{s=0} =$$

$$= (\text{for } n=-1)\ \frac{1}{0!}\left(e^s/(s^2+1)\right)^{(0)}\Big|_{s=0} = \frac{1}{1}*\frac{e^s}{s^2+1}\Big|_{s=0} = 1*\frac{e^0}{0+1} = 1$$

$$= (\text{for } n=0)\ \frac{1}{1!}\left(\frac{e^s}{s^2+1}\right)'\Bigg|_{s=0} = \frac{e^s(s^2+1-2s)}{(s^2+1)^2}\Bigg|_{s=0} = \frac{e^0(0+1-0)}{(0+1)^2} = \frac{1(1)}{1} = 1$$

$$= (\text{for } n=1)\ \frac{1}{2!}\left(\frac{e^s}{s^2+1}\right)''\Bigg|_{s=0} = \frac{1}{2}\left(\frac{e^s(s^2+1-2s)}{(s^2+1)^2}\right)' = \frac{1}{2}\cdot\left(\frac{e^s\left[(s^2+1\cancel{-2s}\cancel{+2s}-2)*(s^2+1)^2-2(s^2+1)(2s)(s^2+1-2s)\right]}{(s^2+1)^4}\right)\Bigg|_{s=0}$$

$$= \frac{1}{2}\left(\frac{e^s(s^2+1)\left[(s^2-1)(s^2+1)-4s(s^2-2s+1)\right]}{(s^2+1)^4}\right)\Bigg|_{s=0} = \frac{1}{2}\left(\frac{e^s(s^4-4s^3+8s^2-4s-1)}{(s^2+1)^3}\right)\Bigg|_{s=0} = \frac{1}{2}*(-1) = -\frac{1}{2}$$

$$= (\text{for } n=2)\ \frac{1}{3!}\left(\frac{e^s}{s^2+1}\right)'''\Bigg|_{s=0} = \frac{1}{6}\left(\frac{e^s(s^4-4s^3+8s^2-4s-1)}{(s^2+1)^3}\right)'\Bigg|_{s=0} = \frac{1}{6}\left(\frac{e^0\left[(0-1+0-4)(0+1)^3-6*0(0+1)^2\cdot e^0(-1)\right]}{(0^2+1)^6}\right) = \frac{-5}{6}$$

$$\therefore \quad \frac{e^z}{z(z^2+1)} = \sum_{-\infty}^{\infty} C_n z^n = \sum_{-\infty}^{-2} \cancel{C_n}^{\,0} z^n + \sum_{-1}^{\infty} C_n z^n = C_{-1}z^{-1}+C_0z^0+C_1z^1+C_2z^2+\cdots$$

$$= 1\cdot z^{-1}+1\cdot z^0+\left(-\frac{1}{2}\right)\cdot z+\left(\frac{-5}{6}\right)z^2+\cdots = \frac{1}{z}+1-\frac{z}{2}-\frac{5z^2}{6}+\cdots, \quad 0<|z|<1$$

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(Note: you can solve also as follows:

$$\frac{e^z}{z(z^2+1)} = \frac{e^z}{z(1-(-z^2))} = \frac{1}{z}(e^z)\left(\frac{1}{1-(-z^2)}\right) = \frac{1}{z}\Big(\underbrace{1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots}_{|z|<\infty}\Big)\Big(\underbrace{1+(-z^2)+(-z^2)^2+\cdots+(-z^2)^k+\cdots}_{|z^2|<1}\Big)$$

$$= \frac{1}{z}\Big(\underbrace{1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots}_{|z|<\infty}\Big)\Big(\underbrace{1-z^2+z^4+\cdots}_{|z|<1}\Big) = \frac{1}{z}\Big(\underbrace{1+(1+0)z+\left(\tfrac{1}{2}+0-1\right)z^2+\left(\tfrac{1}{6}+0-1+0\right)z^3+\cdots}_{(|z|<\infty)\cap(|z|<1)}\Big)$$

$$= \frac{1}{z}\left(1+z-\frac{1}{2}z^2-\frac{5}{6}z^3+\cdots\right) = \frac{1}{z}+1-\frac{z}{2}-\frac{5z^2}{6}+\cdots = \frac{1}{z}+1-\frac{z}{2}-\frac{5z^2}{6}+\cdots$$

$\boxed{\frac{0a}{168}}$

$$f(z) = \csc z =$$

$$= \frac{1}{\sin z} = \left(\text{using Maclaurin for } \sin z\right) \frac{1}{z - z^3/3! + z^5/5! - \cdots}$$

$$= \frac{1}{z} \cdot \frac{1}{[1 + (-z^2/6 + z^4/120 - z^6/7! + \cdots)]} =$$

$$= \frac{1}{z}\left[1 - \left(-\frac{z^2}{6} + \frac{z^4}{120} - \frac{z^6}{7!} \cdots\right) + \left(-\frac{z^2}{6} + \frac{z^4}{120} - \frac{z^6}{7!} + \cdots\right)^2 - \cdots\right]$$

$$= \frac{1}{z}\left[1 + \frac{z^2}{6} + z^4\left(-\frac{1}{120} + \frac{1}{6^2}\right) + z^6\left(\frac{1}{7!} - \frac{2}{720} + \cdots\right) + \cdots\right]$$

$$= \frac{1}{z} + \frac{z}{6} + z^3 \frac{-3+10}{360} + \cdots \quad = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots$$

$\because$ singularity for $\csc z$ is at $\sin z = 0$ i.e $z = k\pi = 0, \pm\pi, \ldots$

$\therefore$ The above Laurent series is convergent for $0 < |z| < \pi$

$\boxed{\dfrac{10b}{168}}$ $f(z) = \dfrac{1}{e^z - 1}$, $0 < |z| < 2\pi$

Consider first the singularities of $f(z)$ at $e^z - 1 = 0$

$\therefore$ $e^z = 1 \angle 2k\pi$ $\therefore$ $z = \text{Log}\,(1 \angle 2k\pi) = \ln 1 + i\,2k\pi = 2k\pi i$

$\therefore$ singularities are at $z = 0, \pm 2\pi i, \pm 4\pi i, \dots, \pm 2k\pi i$

$\therefore$ In the region $0 < |z| < 2\pi$ $f(z)$ is analytic and $z_0 = 0$

$\therefore$ $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = 1 + z + \dfrac{z^2}{2} + \dots + \dfrac{z^n}{n!} + \dots$

$\therefore$ $1 - e^z = 1 - 1 - z - \dfrac{z^2}{2} - \dots - \dfrac{z^n}{n!} - \dots =$

$= -z - \dfrac{z^2}{2} - \dots - \dfrac{z^n}{n!} - \dots = -z\left(1 + \dfrac{z}{2} + \dfrac{z^2}{3!} + \dots \dfrac{z^{n-1}}{n!} + \dots\right)$

$\therefore$ $1 - e^z = -z\,(1 - w)$ where $w = -\dfrac{z}{2} - \dfrac{z^2}{3!} - \dots - \dfrac{z^{n-1}}{n!} = -\sum_{n=2}^{\infty} \dfrac{z^{n-1}}{n!}$

$\therefore$ $f(z) = \dfrac{1}{e^z - 1} = \dfrac{1}{z(1-w)} = \dfrac{1}{z}\left(1 + w + w^2 + \dots + w^n\right) =$

$= \dfrac{1}{z}\left[1 - \left(\dfrac{z}{2} + \dfrac{z^2}{3!} + \dfrac{z^3}{4!} + \dfrac{z^4}{5!} + \dots\right) + \left(\dfrac{z}{2} + \dfrac{z^2}{3!} + \dfrac{z^3}{4!} + \dots\right)^2 - \left(\dfrac{z}{2} + \dfrac{z^2}{3!} + \dots\right)^3 + \left(\dfrac{z}{2} + \dots\right)^4\right]$

$= \dfrac{1}{z}\left[1 - \dfrac{z}{2} - \dfrac{z^2}{6} - \dfrac{z^3}{24} - \dfrac{z^4}{5!} - \dots + \dfrac{z^2}{4} + \dfrac{z^3}{3!} + \dfrac{z^4}{3!^2} + \dfrac{z^4}{4!} + \dots - \dfrac{z^3}{8} - \dfrac{3z^4}{2^2 3!} + \dots + \dfrac{z^4}{2^4} + \dots\right]$

$= \dfrac{1}{z}\left[1 - \dfrac{z}{2} + z^2\left(-\dfrac{1}{6} + \dfrac{1}{4}\right) + z^3\left(-\dfrac{1}{24} + \dfrac{1}{3!} - \dfrac{1}{8}\right) + z^4\left(-\dfrac{1}{5!} + \dfrac{1}{3!^2} + \dfrac{1}{4!} - \dfrac{3}{4 \cdot 3!} + \dfrac{1}{2^4}\right) + \dots + \dots\right]$

$= \dfrac{1}{z}\left[1 - \dfrac{z}{2} + \dfrac{6-4}{24} z^2 + \dfrac{-1+4-3}{24} z^3 + z^4 \dfrac{-4! + 5*4*4 + 5! - 5!*3 + \frac{5!4!}{16}}{5!*4!} + \dots + \dots\right]$

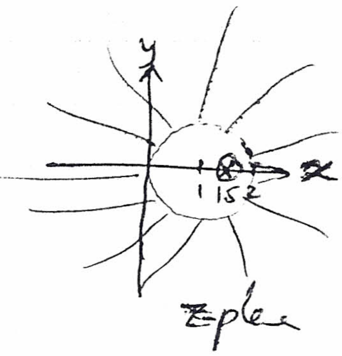
$= \dfrac{1}{z}\left[1 - \dfrac{z}{2} + \dfrac{2}{24} z^2 + 0 z^3 + z^4 \cdot \dfrac{-4*3*2 + 80 + 5*4*3*2 - 5*4*3^2*2 + \frac{5*4^2*3^2*2^2}{16}}{5*4^2*3^2*2^2} + \dots\right]$

$= \dfrac{1}{z}\left[1 - \dfrac{z}{2} + \dfrac{z^2}{12} + z^4 \cdot \dfrac{-24 + 80 + 120 - 360 + 180}{5*4^2*3^2*2^2} + \dots\right]$ 85

$= \dfrac{1}{z} - \dfrac{1}{2} + \dfrac{z}{12} + z^3 \cdot \dfrac{(-4)}{720*4} + \dots = \dfrac{1}{z} - \dfrac{1}{2} + \dfrac{z}{12} - \dfrac{z^3}{720} + \dots$

$\therefore$ OK.

\# $f(z) = \dfrac{1}{2z-3}$, $|z-1| > 1$



$\because$ Singularity at $z = 1.5$ is out of region.

$\therefore$ $f(z)$ is analytic in $|z-1| > 1$

$\therefore$ Laurent Series, with $z_0 = 1$. With short cut, $\left|\dfrac{1}{z-1}\right| < 1$

$$\therefore \frac{1}{2z-3} = \frac{1}{2z-2-1} = \frac{1}{2(z-1)-1} = \frac{1}{2(z-1)} \cdot \frac{1}{1-\frac{1}{2(z-1)}}$$

$$= \frac{1}{2(z-1)} \cdot \left[1 + \frac{1}{2(z-1)} + \frac{1}{2^2(z-1)^2} + \cdots + \frac{1}{2^n(z-1)^n} + \cdots\right]$$

$$= \frac{1}{2(z-1)} \cdot \sum_{k=0}^{\infty} 2^{-k}(z-1)^{-k} = \sum_{k=1}^{\infty} 2^{-k}(z-1)^{-k}$$

for convergence: $|(z-1)^{-1}| < 1$ i.e $|z-1| > 1$ (OK

\# $f(z) = \dfrac{1}{2z-3} =$

$$= \frac{1}{2(z-2)+1} = \frac{1}{2(z-2)\left(1+\frac{1}{2(z-2)}\right)} =$$

$$= \frac{1}{2(z-2)} \cdot \left[1 - \frac{1}{2(z-2)} + \frac{1}{4(z-2)^2} - \frac{1}{8(z-2)^3} + \cdots\right]$$

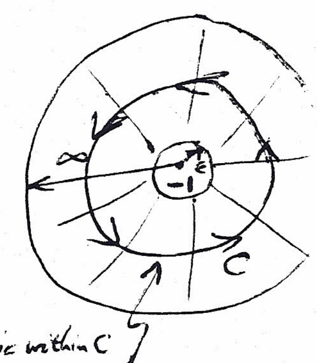
$$= \frac{1}{2(z-2)} \cdot \sum_{k=0}^{\infty} (-1)^k \left[2(z-2)\right]^{-k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \left[2(z-2)\right]^{-k-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}(z-2)^{k+1}}.$$

Convergent for $\left|\dfrac{1}{z-2}\right| < 1$ or $|z-2| > 1$

$\boxed{\frac{1b}{179}}$ $\frac{z^2}{1+z}$, singularity at $z_0=-1$

Using Laurent expansion about $z_0=-1$ in the annulus shown:



$\therefore \frac{z^2}{1+z} = \sum_{-\infty}^{\infty} C_n(z-z_0)^n = \sum_{n=-\infty}^{\infty} C_n (z+1)^n$

$C_n = \frac{1}{2\pi i} \oint_C \frac{s^2/(1+s)}{(s+1)^{n+1}}\, ds = \frac{1}{2\pi i}\oint_C \frac{s^2\,ds}{(1+s)^{n+2}}$

$=$ (for $n<-1$) 0 because the integrand is then analytic within C

C: $\epsilon<|z+1|<\infty$

$=$ (for $n\geq -1$) $\frac{1}{2\pi i} * 2\pi i \frac{\left.(s^2)^{(n+1)}\right|_{s=-1}}{(n+1)!} =$

$= \frac{1}{(n+1)!} \left.(s^2)^{(n+1)}\right|_{s=-1} =$

$=$ (for $n=-1$) $\frac{1}{0!} \left.(s^2)^{(0)}\right|_{s=-1} = \frac{1}{1}\left.(s^2)\right|_{-1} = 1*(-1)^2 = 1$

$=$ (for $n=0$) $\frac{1}{1!}\left.(s^2)'\right|_{s=-1} = 1\left(\left.2s\right|_{s=-1}\right) = -2$

$=$ (for $n=1$) $\frac{1}{2!}\left.(s^2)''\right|_{-1} = \frac{1}{2} * 2 = 1$

$=$ (for $n\geq 2$) $\frac{1}{(n+1)!}\left.(s^2)^{(n+1)}\right|_{-1} = 0$

$\therefore \frac{z^2}{1+z} = \sum_{-\infty}^{\infty} C_n (z+1)^n = \sum_{-\infty}^{-2} \cancel{C_n}^{0}(z+1)^n + \sum_{-1}^{1} C_n(z+1)^n + \sum_{2}^{\infty}\cancel{C_n}^{0}(z+1)^n$

$= \sum_{-1}^{1} C_n (z+1)^n = C_{-1}(z+1)^{-1} + C_0(z+1)^0 + C_1(z+1)^1 = 1(z+1)^{-1} - 2(z+1)^0 + 1(z+1)^1$

$= \frac{1}{z+1} - 2 + (z+1)$ at $0<|z+1|<\infty$

$\therefore$ The principal part of $\frac{z^2}{1+z}$ is $\frac{1}{z+1}$ at its singularity point $z=-1$
the singularity is a simple pole.

$\boxed{\frac{2a}{179}}$

$$\frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$\therefore\ z+1 = A(z-2) + Bz \qquad \therefore\ 1 = -2A \qquad \therefore\ A = -\frac{1}{2}$

& $1 = A + B \qquad \therefore\ B = 1 - A = \frac{3}{2}$

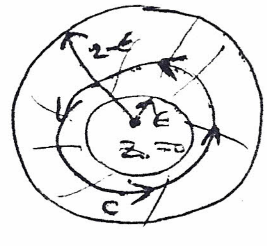
$$\therefore\ \frac{z+1}{z^2-2z} = \frac{-1/2}{z} + \frac{3/2}{z-2}$$

$\therefore$ Singularity of $\frac{z+1}{z^2-2z}$ is at $z=0$ & $z=2$ and both are simple poles with residues $-\frac{1}{2}$ & $\frac{3}{2}$ respectively.

(Note: you can also solve by Laurent expansion as follows

$\frac{z+1}{z^2-2z}$ has singularity at $z_0^2 - 2z_0 = 0 \Rightarrow z_0 = 0$ or 2

1. singularity at $z_0 = 0$ expand at annulus shown.



$0<|z|<2$

$$\therefore\ \frac{z+1}{z^2-2z} = \sum_{-\infty}^{\infty} C_n (z-0)^n = \sum_{-\infty}^{\infty} C_n z^n$$

$$C_n = \frac{1}{2\pi i}\oint_C \frac{(s+1)/(s^2-2s)}{s^{n+1}}\,ds = \frac{1}{2\pi i}\oint_C \frac{(s+1)/(s-2)}{s^{n+2}}\,ds$$

$= (\text{for } n < -1)\ 0$ because the integrand is analytic within C

$= (\text{for } n \geq -1)\ \frac{1}{\cancel{2\pi i}} * \frac{\cancel{2\pi i}}{(n+1)!}\left(\frac{s+1}{s-2}\right)^{(n+1)}\Big|_{s=0}$ (Cauchy Theo)

$= (\text{for } n=-1)\ \frac{1}{0!}\left(\frac{s+1}{s-2}\right)^{(0)}\Big|_0 = \frac{1}{1}\left(\frac{s+1}{s-2}\Big|_0\right) = 1 * \frac{1}{-2} = -\frac{1}{2}$

$= (\text{for } n=0)\ \frac{1}{1!}\left(\frac{s+1}{s-2}\right)'\Big|_0 = 1\left(\frac{s-2-(s+1)}{(s-2)^2}\right)\Big|_0 = \frac{-3}{(s-2)^2}\Big|_0 = \frac{-3}{4}$

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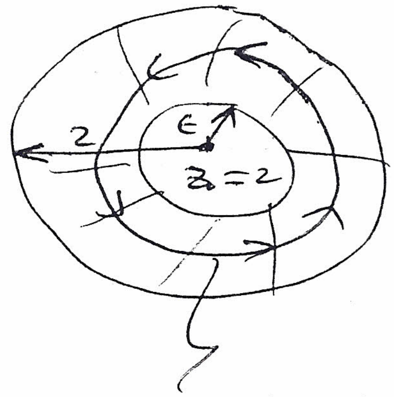
$= (\text{for } n=1)\ \frac{1}{2!}\left(\frac{s+1}{s-2}\right)''\Big|_0 = \frac{1}{2}\left(\frac{-3}{(s-2)^2}\right)'\Big|_0 = \frac{1}{2}\left(\frac{6}{(s-2)^3}\right)\Big|_0 = \frac{1}{2}\,\frac{6}{-8} = \frac{-3}{8}$

$$\therefore\ \frac{z+1}{z^2-2z} = \sum_{-\infty}^{\infty} C_n z^n = \sum_{-1}^{\infty} C_n z^n = C_{-1} z^{-1} + C_0 z^0 + \cdots = -\frac{1}{2} z^{-1} - \frac{3}{4} z^0 - \frac{3}{8} z^1 + \cdots$$

$= \frac{-1/2}{z} - \frac{3}{4} - \frac{3z}{8} + \cdots \qquad \therefore$ pole of order 1 (simple) residue $= -\frac{1}{2}$

2 | singularity at $z_0 = 2$, expand at the annulus shown.

$$\therefore \frac{z+1}{z^2-2z} = \sum_{-\infty}^{\infty} C_n (z-z_0)^n$$



$C: 0 < |z-2| < 2$

$$= \sum_{-\infty}^{\infty} C_n (z-2)^n$$

$$C_n = \frac{1}{2\pi i} \oint_C \frac{\{(s+1)/(s^2-2s)\}\, ds}{(s-2)^{n+1}} =$$

$$= \frac{1}{2\pi i} \oint_C \frac{(s+1)/s}{(s-2)^{n+2}} \cdot ds$$

$= (\text{for } n < -1)\ 0$ (integrand analytic within C)

$$= (\text{for } n \geq -1)\ \frac{1}{\cancel{2\pi i}} \times \frac{\cancel{2\pi i}}{(n+1)!} \left(\frac{s+1}{s}\right)^{(n+1)} \Bigg|_{s=2} =$$

$$= (\text{for } n = -1)\ \frac{1}{0!} \left(1 + \frac{1}{s}\right)^{(0)} \Bigg|_{s=2} = \frac{1}{1} \left(1 + \frac{1}{s}\Big|_{s=2}\right) = 1 \left(1 + \frac{1}{2}\right) = \frac{3}{2}$$

$$= (\text{for } n = 0)\ \frac{1}{1!} \left(1 + \frac{1}{s}\right)^{(1)} \Bigg|_{s=2} = 1 \left(1 + \frac{1}{s}\right)' \Bigg|_2 = \left(0 - \frac{1}{s^2}\right)\Bigg|_2 = -\frac{1}{4}$$

$$= (\text{for } n = 1)\ \frac{1}{2!} \left(1 + \frac{1}{s}\right)'' = \frac{1}{\cancel{2}} \cdot \frac{+\cancel{2}}{s^3}\Bigg|_2 = \frac{1}{8}$$

$$\therefore \frac{z+1}{z^2-2z} = \sum_{-1}^{\infty} C_n (z-2)^n = C_{-1} (z-2)^{-1} + C_0 (z-2)^0 + C_1 (z-2)^1 + \cdots$$

$$= \frac{3}{2}(z-2)^{-1} - \frac{1}{4}(z-2)^0 + \frac{1}{8}(z-2)^1 + \cdots$$

$$= \frac{3/2}{z-2} - \frac{1}{4} + \frac{1}{8}(z-2) + \cdots \qquad 0 < |z-2| < 2$$

$\therefore$ Pole of order one (simple) and residue $= 3/2$ $\quad \therefore$ OK)

—

$\boxed{\frac{36}{180}}$ $z^{-3}\csc(z^2) = \dfrac{1}{z^3\sin(z^2)}$ = (using expansion of $\sin w = w - \frac{w^3}{3!} + \frac{w^5}{5!} - -$)

$$= \frac{1}{z^3}\cdot\frac{1}{z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots} = \frac{1}{z^5}\cdot\frac{1}{\left(1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots\right)} =$$

= (using the expansion $\frac{1}{1-w} = 1 + w + w^2 + w^3 + - - - -$)

$$= \frac{1}{z^5}\cdot\left(1 + \left(\frac{z^4}{3!} - \frac{z^8}{5!} + \cdots\right) + \left(\frac{z^4}{3!} - \frac{z^8}{5!} + \cdots\right)^2 + \cdots\right) =$$

$$= \frac{1}{z^5} + \frac{1}{6z} + \left(-\frac{1}{5!} + \frac{1}{(3!)^2}\right)z^3 + \cdots = \frac{1}{z^5} + \frac{1/6}{z} + \frac{7}{360}z^3 + \cdots$$

$\therefore$ The residue at $z_0 = 0$ of $z^{-3}\csc(z^2)$ is $\frac{1}{6}$,
The pole is of order 5

(Note: you can not solve by Laurent series in the annulus shown about $z_0 = 0$, because then:

$$z^{-3}\csc(z^2) = \sum_{-\infty}^{\infty} C_n z^n$$

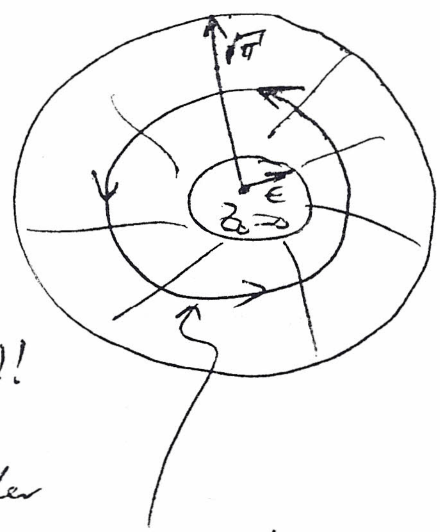
$$\& \; C_n = \frac{1}{2\pi i}\oint_C \frac{s^{-3}\csc(s^2)}{s^{n+1}}\,ds =$$

$$= \frac{1}{2\pi i}\oint_C \frac{\csc(s^2)}{s^{n+4}}\,ds \quad \text{, tedious integration !!}$$

However, you can inspect the pole order as follows

$$z^{-3}\csc(z^2) = \frac{1}{z^3\sin(z^2)} = \left(\text{since at } z \to 0 \; \sin z^2 \approx z^2\right)$$

$= (\text{near } z=0)\ \frac{1}{z^5}$ $\therefore$ pole of order 5 $\therefore$ OK).



$C: 0 < |z| < \sqrt{\pi}$

$\sin(\sqrt{\pi})^2 = 0 \quad \therefore \csc(\sqrt{\pi})^2 = \infty$

$\boxed{\frac{3c}{180}}$ $f(z) = z\cos\frac{1}{z}$

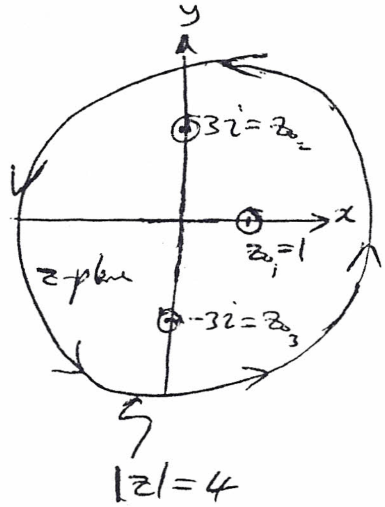
$\because \cos w = 1 - \frac{w^2}{2} + \frac{w^4}{4!} + \cdots\cdots (-)^n \frac{w^{2n}}{(2n)!}$ putting $w = \frac{1}{z}$

$\therefore f(z) = z\cos\frac{1}{z} = z\left(1 - \frac{(1/z)^2}{2} + \frac{(1/z)^4}{4!} \cdots (-)^n \frac{(1/z)^{2n}}{(2n)!} + \cdots\right)$

$= z - \frac{1}{2z} + \frac{1}{4!z^3} + \cdots\cdots (-)^n \frac{1}{z^{2n-1}(2n)!} + \cdots$

$\therefore$ The singularity is essential at $z_0 = 0$ & the residue at it $= -1/2$

$\boxed{\frac{4b}{180}}$ $\oint_C \frac{3z^2+2}{(z-1)(z^2+9)} \cdot dz = I$ $\quad C: |z| = 4$



Let $\frac{3z^2+2}{(z-1)(z^2+9)} = f(z)$

$\therefore f(z)$ has singularites at $z_0 = 1, \pm i3$

$\therefore \oint_C f(z)\,dz = \oint_{C_1} + \oint_{C_2} + \oint_{C_3}$ of $f(z)\,dz$

where $C_1: |z-1| = \epsilon_1$, $C_2: |z-3i| = \epsilon_2$, $C_3: |z+3i| = \epsilon_3$

$\therefore I = \oint_{C_1} \frac{(3z^2+2)/(z^2+9)}{z-1}\,dz + \oint_{C_2} \frac{(3z^2+2)/[(z-1)(z+3i)]}{z-3i}\,dz + \oint_{C_3} \frac{(3z^2+2)/[(z-1)(z-3i)]}{z+3i}\,dz$

$= 2\pi i * (3z^2+2)/(z^2+9)\Big|_{z=1} + 2\pi i * (3z^2+2)/[(z-1)(z+3i)]\Big|_{3i} + 2\pi i * (3z^2+2)/[(z-1)(z-3i)]\Big|_{-3i}$

$= 2\pi i * \frac{3+2}{1+9} + 2\pi i * \frac{-27+2}{(3i-1)6i} + 2\pi i * \frac{-27+2}{(-3i-1)(-6i)} =$

$= 2\pi i\left[\frac{5}{10} - \frac{25(3i+1)}{(-9-1)6i} + \frac{25(3i-1)}{(-9-1)(-6i)}\right] =$

$= 2\pi i\left\{\frac{1}{2} - \frac{25(3i+1+3i-1)}{(-10)6i}\right\} = 2\pi i\left\{\frac{1}{2} + \frac{25}{60i} * 6i\right\} =$

$= 2\pi i * \left[\frac{1}{2} + \frac{25}{10}\right] = 2\pi i\left(\frac{1}{2} + \frac{5}{2}\right) = 6\pi i$

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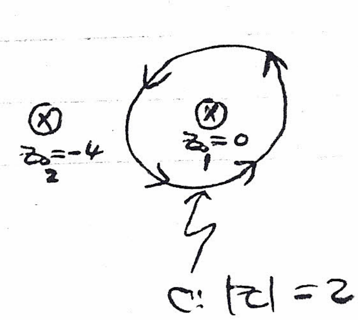
$\boxed{\frac{5a}{180}}$

$$\oint_C \frac{dz}{z^3(z+4)} = (\text{singularities at } z_0 = 0, -4)$$



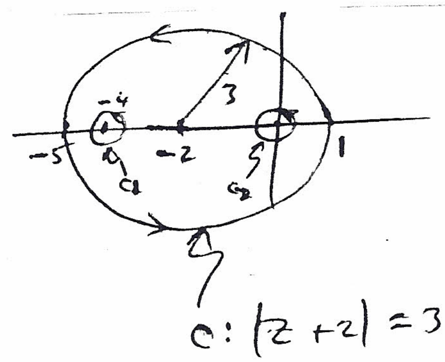
$$= \oint_C \frac{1/(z+4)}{z^3}\, dz = \frac{[1/(z+4)]''}{2!}\Big|_0 * 2\pi i =$$

$$= \frac{2\pi i}{2}\left(-1(z+4)^{-2}\right)'\Big|_0 = \pi i * \left(2(z+4)^{-3}\Big|_0\right) =$$

$$= \pi i * 2(4)^{-3} = \frac{2\pi i}{4^3} = \underline{\frac{2\pi i}{64}} = \pi i/32$$

$\boxed{\frac{5b}{180}}$

$$I = \oint_{C: |z+2|=3} \frac{dz}{z^3(z+4)} =$$



$$= \oint_{C_1 + C_2} \frac{dz}{z^3(z+4)} =$$

$$= \oint_{C_1: |z+4|=\epsilon} \frac{(1/z^3)\,dz}{z+4} + \oint_{C_2: |z|=\epsilon} \frac{[1/(z+4)]\,dz}{z^3} =$$

$$= 2\pi i \left(\frac{1}{z^3}\Big|_{-4}\right) + 2\pi i \left(\frac{1}{z+4}\right)''\Big|_0 / 2! = 2\pi i * \left(\frac{1}{-64}\right) + 2\pi i\, \frac{(-1)(-2)}{(z+4)^3}\Big|_0 \cdot \frac{1}{2!}$$

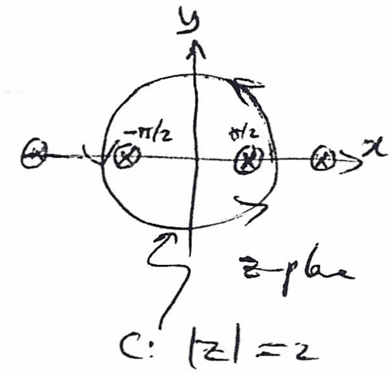
$$= (2\pi i)\left[-\frac{1}{64} + \frac{\cancel{2}}{4^3 * \cancel{2!}}\right] = 2\pi i \cdot \left(\frac{-1+1}{64}\right) = 0$$

$$\therefore\ I = \oint_C \frac{dz}{z^3(z+4)} = \underline{0}$$

$\boxed{\dfrac{69}{180}}$

$$\oint_C \tan z\,dz = \oint_C \frac{\sin z}{\cos z}\,dz$$

(singularity at $z_0 = \left(\frac{2k+1}{2}\right)\pi$, k integer & $z_0 = \pm\pi/2$ lie within C)



$$= \oint_{C_1}\frac{\sin z}{\cos z}\,dz + \oint_{C_2}\frac{\sin z}{\cos z}\,dz$$

(where $C_1: \left|z-\frac{\pi}{2}\right| = \epsilon_1$, $C_2: \left|z+\frac{\pi}{2}\right| = \epsilon_2$)

$$= \oint_{C_1}\frac{\sin z\,dz}{\cos z\Big|_{\frac{\pi}{2}} + (\cos z)'\Big|_{\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right) + (\cos z)''\Big|_{\pi/2}\frac{\left(z-\frac{\pi}{2}\right)^2}{2}+\cdots} + \oint_{C_2}\frac{\sin z\,dz}{\cos z\Big|_{-\pi/2} + (\cos z)'\Big|_{-\pi/2}\left(z+\frac{\pi}{2}\right) + (\cos z)''\Big|_{-\pi/2}\frac{(z+\pi/2)^2}{2}+\cdots}$$

$$= \oint_{C_1}\frac{\sin z\,dz}{0-(z-\pi/2)-0+\frac{(z-\pi/2)^3}{3!}\cdots} + \oint_{C_2}\frac{\sin z\,dz}{0+(z+\pi/2)-0-\frac{(z+\pi/2)^3}{3!}\cdots}$$

$$= \oint_{C_1}\frac{\sin z\,dz}{-(z-\pi/2)\left[1-\frac{(z-\pi/2)^2}{3!}+\cdots\right]} + \oint_{C_2}\frac{\sin z\,dz}{(z+\pi/2)\left[1-\frac{(z+\pi/2)^2}{3!}+\cdots\right]} =$$

(since $\frac{1}{1-w} = 1+w+w^2+\cdots$)

$$= \oint_{C_1}\frac{\sin z\,dz}{-(z-\pi/2)}\left[1+\frac{(z-\pi/2)^2}{3!}+\cdots+\left(\frac{(z-\pi/2)^2}{3!}+\cdots\right)^2+\cdots\right] + \oint_{C_2}\frac{\sin z\,dz}{(z+\pi/2)}\left[1+\frac{(z+\pi/2)^2}{3!}\cdots+\left(\frac{(z+\pi/2)^2}{3!}\right)^2\right]$$

$$= \oint_{C_1}\frac{\sin z\,dz}{-(z-\pi/2)} + \underbrace{\cancel{\oint_{C_1}\frac{(z-\pi/2)}{3!}\sin z\,dz}}_{\text{analytic within } C_1 \;\to\; 0}+\cdots + \oint_{C_2}\frac{\sin z\,dz}{z+\pi/2} + \underbrace{\cancel{\oint_{C_2}\frac{(z+\pi/2)}{3!}\sin z\,dz}}_{\text{analytic within } C_2}+\cdots$$

$$= \oint_{C_1}\frac{-\sin z\,dz}{z-\pi/2} + \oint_{C_2}\frac{\sin z\,dz}{z+\pi/2} = 2\pi i * \left(-\sin z\Big|_{\pi/2}\right) + 2\pi i\left(\sin z\Big|_{-\pi/2}\right) =$$

$$= 2\pi i(-1) + 2\pi i(-1) = -4\pi i$$

(Note: you can also solve by finding residues of $\tan z$ in C_1 & C_2

$\therefore B_1 =$ (since $\tan z$ is a simple pole at $z_0 = \frac{\pi}{2}$) $\dfrac{\phi^{(1-1)}(\pi/2)}{(1-1)!} = \lim_{z\to\pi/2}\dfrac{(z-\pi/2)\cdot\tan z}{0!} =$

$= \lim_{z\to\pi/2}\dfrac{(z-\pi/2)}{\cot z} = \left(\frac{0}{0}\text{ use L'Hospital rule}\right)\lim_{z\to\pi/2}\dfrac{1}{-\csc^2 z} = -1$

& $B_2 = \lim_{z\to-\pi/2}\dfrac{(z+\pi/2)\tan z}{0!} = \lim_{z\to-\pi/2}\dfrac{z+\pi/2}{\cot z} = \lim_{z\to-\frac{\pi}{2}}\dfrac{1}{-\csc^2 z} = -1$

$\therefore \oint \tan z\,dz = 2\pi i\sum B = 2\pi i(-1-1) = -4\pi i$, $\therefore$ ok).

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$\boxed{\frac{6b}{180}}$ Let $\oint_{C:|z|=2} \frac{dz}{\sinh 2z} = I.$ singularities of $\sinh 2z$ is at $\sinh 2z = 0$ $\therefore \sin\frac{2z}{i} = 0$

$\therefore \frac{2z}{i} = k\pi$ $\therefore z = k\pi i/2$, only $0, \pm\frac{\pi i}{2}$ are in C.

$$\therefore I_1 = \oint_{C_1:|z|=\epsilon} \frac{dz}{2z\left(1+\frac{4z^2}{6}+\frac{16z^4}{120}+\cdots\right)} = \oint_{C_1} \frac{1}{2z}\left[1-\left(\frac{4z^2}{6}+\frac{16z^4}{120}+\cdots\right)+\left(\frac{4z^2}{6}+\cdots\right)^2+\cdots\right]dz$$

$$= \oint_{C_1}\left(\frac{1}{2z}-\frac{2z}{6}-\frac{8z^3}{120}\text{-----}+\frac{8z^3}{36}+\text{---}\quad\text{----}+\cdots\right)dz =$$

$$= \frac{1}{2} * 1 * 2\pi i - 0 - 0 + \text{---} + 0 + \text{---} \quad -0 \text{--} +0\text{---} = \pi i$$

$\therefore \oint_{C_1} \frac{dz}{\sinh 2z} = \pi i$ (Note: Residue of $\frac{1}{\sinh 2z}$ at 0 is $\left.\frac{z}{\sinh 2z}\right|_0 = \frac{1}{2\cosh 2z}$ at 0

$= \frac{1}{2}$ and hence $\oint_{C_1} \frac{dz}{\sinh 2z} = 2\pi i * \frac{1}{2} = \pi i$ $\therefore$ OK)

$$+ I_2 = \oint_{C_2: |z-\frac{\pi i}{2}|=\epsilon} \left[1/(\sinh 2z)\right]dz = 2\pi i * \left(\operatorname{Res} \left.\frac{1}{\sinh 2z}\right|_{z=\pi i/2}\right) = 2\pi i * \left.\left(\frac{z-\pi i/2}{\sinh 2z}\right)\right|_{\pi i/2}$$

$$= 2\pi i * \left.\frac{1}{2\cosh 2z}\right|_{z=\pi i/2} = 2\pi i * \frac{1}{2\cosh \pi i} = \pi i * \frac{1}{\cos\pi} = -\pi i$$

$$+ I_3 = \oint_{C_3: |z+\pi i/2|=\epsilon} (1/\sinh 2z)\,dz = 2\pi i * \left.\frac{z+\pi i/2}{\sinh 2z}\right|_{-\pi i/2} = 2\pi i * \left.\frac{1}{2\cosh 2z}\right|_{-\pi i/2} =$$

$$= \pi i * \frac{1}{\cosh(-\pi i)} = \pi i * \frac{1}{\cos -\pi} = \pi i * (-1) = -\pi i$$

$$\therefore I = \oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} = I_1 + I_2 + I_3 = \pi i + (-\pi i) + (-\pi i) = -\pi i$$

$$\therefore \oint_{C:|z|=2} \frac{dz}{\sinh 2z} = -\pi i$$

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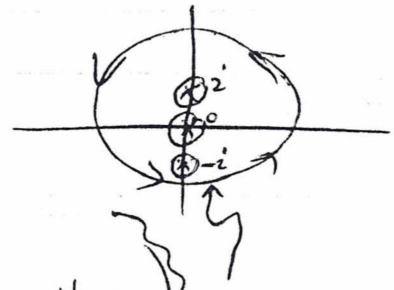
$\boxed{\dfrac{6C}{180}}$

$$\oint_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$$

(singularities at $z_0 = 0, \pm i$ all within C)



$$\therefore \oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3}$$

(where $C_1 : |z| = \epsilon_1$, $C_2 : |z - i| = \epsilon_2$, $C_3 : |z + i| = \epsilon_3$) $C : |z| = 2$

$$\therefore \oint_C \frac{\cosh \pi z}{z(z^2+1)}\, dz = \oint_{C_1} \frac{\cosh \pi z / (z^2+1)}{z}\, dz + \oint_{C_2} \frac{\cosh \pi z / [(z+i) z]}{(z-i)}\, dz + \oint_{C_3} \frac{\cosh \pi z / [(z-i) z]}{z+i}\, dz$$

$$= 2\pi i \left[\left. \frac{\cosh \pi z}{z^2+1} \right|_0 + \left. \frac{\cosh \pi z}{z(z+i)} \right|_i + \left. \frac{\cosh \pi z}{z(z-i)} \right|_{-i} \right] = 2\pi i \left[\frac{1}{1} + \frac{\cos \pi}{-2} + \frac{\cos \pi}{-2} \right] =$$

$$= 2\pi i \left(1 + \tfrac{1}{2} + \tfrac{1}{2} \right) = 4\pi i$$

95

$\boxed{\frac{7c}{180}}$ $I = \oint_{C: |z|=1} z^{-2} \csc z \, dz = \oint_C \frac{1}{z^2 \sin z} dz$

singularities at 0 & $\sin z = 0$ i.e $z = k\pi = 0, \pm\pi, \pm 2\pi$

$\therefore$ The only singularity within C is at $z=0$ of order 3.

$$\therefore \quad I = 2\pi i \operatorname{Res}_{\text{order } 3}\left(\frac{1}{z^2 \sin z}\right)\Big|_{\text{at } 0} = 2\pi i \left(\frac{z^3}{z^2 \sin z}\right)''\Big|_{\text{at } 0} \Big/ 2! = \pi i \left(\frac{z}{\sin z}\right)''\Big|_0 =$$

$$= \pi i \left(\frac{\sin z - z\cos z}{\sin^2 z}\right)'\Big|_0 = \pi i \left(\frac{(\cos z - \cos z + z\sin z)\sin^2 z - \sin 2z\,(\sin z - z\cos z)}{\sin^4 z}\right)\Big|_0 =$$

$$= \pi i \left(\frac{z\sin^3 z - \sin 2z\,(\sin z - z\cos z)}{\sin^4 z}\right)\Big|_0 = \pi i \left(\frac{z^4 - 2z\left(z - \frac{z^3}{6} - z\left(1 - \frac{z^2}{2}\right)\right)}{z^4}\right) =$$

$$= \pi i\left(1 - 2\left(-1/6 + 1/2\right)\right) = \pi i\,(1 - 2/3) = \pi i/3$$

OR $\quad \oint_C \frac{1}{z^2 \sin z} dz = \oint_C \frac{1}{z^2(z - z^3/6 + z^5/5! + \cdots)} dz =$

$$= \oint_C \frac{1}{z^3(1 - z^2/6 + z^4/5! + \cdots)} dz =$$

$$= \oint_C \frac{1}{z^3}\left[1 + \frac{z^2}{6} - \frac{z^4}{5!} + \cdots\right] dz = \oint_C \left(\frac{1}{z^3} + \frac{1}{6z} + \cdots\right) dz$$

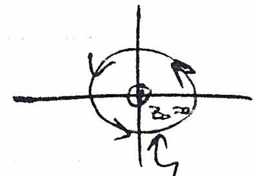
$$= 2\pi i * \left[\frac{(1)''}{2!} + \left(\frac{1}{6}\right)^{(0)} \Big/ 0! + 0 + 0 \cdots\right] = 2\pi i\left(0 + \frac{1}{6}\right) = \frac{\pi i}{3}$$

$$\therefore \quad \oint_{|z|=1} z^{-2} \csc z \, dz = \pi i/3$$

$\boxed{\dfrac{7d}{180}}$

$$\oint_C z \exp\left(\tfrac{1}{z}\right) dz =$$

$$= \oint_C z\left(e^{\frac{1}{z}}\right) dz = \oint_C z\left(e^{w}\right) dz \quad , w = \tfrac{1}{z}$$



C: $|z|=1$

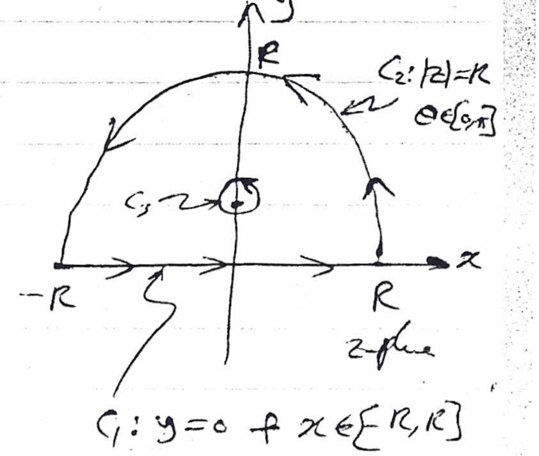
$$= \oint_C z\left(1 + w + \frac{w^2}{2!} + \cdots + \frac{w^n}{n!} + \cdots\right) dz$$

$$= \oint_C z\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots \frac{1}{n!\,z^n} + \cdots\right) dz$$

$$= \oint_C \left(z + 1 + \frac{1}{2z} + \frac{1}{6z^2} + \cdots + \frac{1}{n!\,z^{n-1}}\right) dz \quad = 2\pi i * \text{residue}$$

$$= 2\pi i * \left(\frac{1}{2}\right) = \pi i$$

$\boxed{\frac{1}{186}}$ Let $I = \int_0^\infty \frac{dx}{1+x^2}$



$$\therefore\ 2I = 2\int_0^\infty \frac{dx}{1+x^2} = \int_{-\infty}^\infty \frac{dx}{1+x^2}$$

$$= \lim_{R\to\infty} \int_{-R,\,C_1}^{R} \frac{dz}{1+z^2}$$

$$= \lim_{R\to\infty}\left(\oint_{C_1+C_2} - \int_{C_2}\right)\frac{dz}{1+z^2}$$

Singularities at $z^2=-1$ or $z=\pm i$ only $+i$ is within C_1+C_2

$$\therefore \oint_{C_1+C_2} \frac{dz}{z^2+1} = \oint_{C_3: |z-i|=\epsilon} dz/(z^2+1) = 2\pi i \operatorname{Res} \frac{1}{z^2+1}\Big|_i = 2\pi i\left(\frac{z-i}{z^2+1}\right)\Big|_i =$$

$$= 2\pi i\left(\frac{1}{2z}\right)\Big|_i = 2\pi i/2i = \pi$$

$$\&\ \left|\int_{C_2}\frac{dz}{1+z^2}\right| \le \int\left|\frac{dz}{1+z^2}\right| \le \int\frac{|dz|}{|1-|z|^2|} = \int\frac{|dz|}{|1-R^2|} =$$

$$= \int\frac{|dz|}{R^2-1} = \frac{1}{R^2-1}\int|dz| = \frac{1}{R^2-1}\cdot\pi R = \frac{\pi R}{R^2-1} \to 0 \text{ as } R\to\infty$$

$$\therefore \lim_{R\to\infty}\int_{C_2}\frac{dz}{1+z^2} = 0$$

$$\therefore\ 2I = \lim_{R\to\infty}\left(\oint_{C_1+C_2} - \int_{C_2}\right)\frac{dz}{1+z^2} = \lim_{R\to\infty}\pi - \lim_{R\to\infty}\int_{C_2}\frac{dz}{z^2+1} = \pi - 0$$

$\therefore\ 2I = \pi \quad \therefore\ I = \pi/2$

$$\therefore \int_0^\infty \frac{dx}{1+x^2} = \pi/2$$

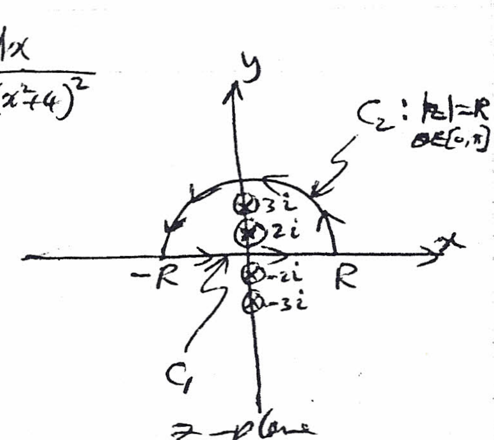
$\boxed{\frac{5}{186}}$

$$I=\int_0^\infty \frac{x^2dx}{(x^2+9)(x^2+4)^2} = \text{(because the integrand is even function)}$$

$$= \frac12\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+9)(x^2+4)^2} = \frac12 \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2dx}{(x^2+9)(x^2+4)^2}$$

$$= \frac12 \lim_{R\to\infty}\int_{z=-R,\,C_1}^{z=R}\frac{z^2dz}{(z^2+9)(z^2+4)^2}$$

$$=\frac12\lim_{R\to\infty}\left[\oint_{C_1+C_2}\frac{z^2dz}{(z^2+9)(z^2+4)^2} - \int_{z=-R,\,C_2}^{z=R}\frac{z^2dz}{(z^2+9)(z^2+4)^2}\right]$$



Note: you can choose C_2 to be the negative half circle and the answer will be the same due to symmetry of poles.

$$=\frac12\left[\oint_{C_3}+\oint_{C_4} - \lim_{R\to\infty}\int_{z=-R,\,C_2}^{R}\frac{z^2dz}{(z^2+9)(z^2+4)^2}\right]$$

(where $C_3: |z-2i|=\epsilon_1$, $C_4: |z-3i|=\epsilon_2$)

$$\oint_{C_3} = \oint_{C_3}\frac{z^2/[(z^2+9)(z+2i)^2]}{(z-2i)^2}dz = \frac{2\pi i}{1!}\left(\frac{z^2}{(z^2+9)(z+2i)^2}\right)'\Big|_{z=2i} =$$

$$=2\pi i\left(\frac{2z}{(z^2+9)(z+2i)^2} - \frac{2z^3}{(z^2+9)^2(z+2i)^2} - \frac{2z^2}{(z^2+9)(z+2i)^3}\right)\Big|_{z=2i}=$$

$$=2\pi i\left(\frac{4i}{(-4+9)(-16)} - \frac{2(-8i)}{(-4+9)^2(-16)} - \frac{2(-4)}{(-4+9)(-64i)}\right) = 2\pi i\left(\frac{5*4i(-4i)+16i(-4i)+8(-5)}{5^2*64i}\right)=$$

$$=2\pi i*\frac{80+64-40}{1600i} = \frac{\pi}{800}(104) = \frac{26\pi}{200}$$

& $$\oint_{C_4} = \oint_{C_4}\frac{z^2/[(z^2+4)^2(z+3i)]}{(z-3i)}dz = 2\pi i * \frac{z^2}{(z^2+4)^2(z+3i)}\Big|_{z=3i} = 2\pi i\frac{-9}{(-9+4)^2(6i)}=$$

$$=2\pi i*\frac{-9}{25*6i} = \frac{-18\pi i}{150i} = \frac{-18\pi}{150} = \frac{-6\pi}{50} = \frac{-24\pi}{200}$$

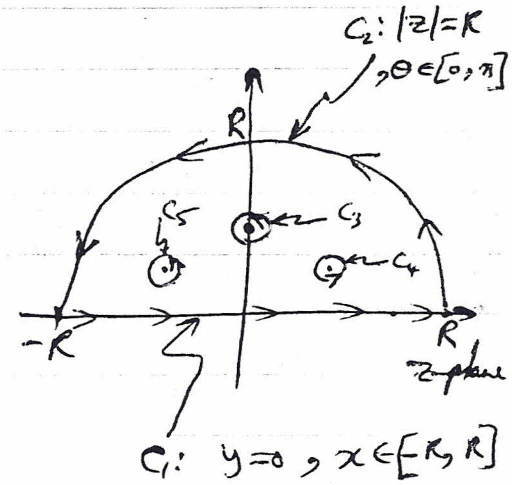
& $$\left|\int_{z=-R,\,C_2}^{R}\frac{z^2dz}{(z^2+9)(z^2+4)^2}\right| \le \int_{z=-R,\,C_2}^{R}\frac{|z^2||dz|}{||z^2|-|9||*||z^2|-|4||^2} = \frac{R^2*\pi R}{(R^2-9)(R^2-4)^2} = \frac{\pi R^3}{(R^2-9)(R^2-4)^2}$$

$$\underset{\text{as } R\to\infty}{=}\frac{\pi/R^3}{(1-9/R^2)(1-4/R^2)^2} = \frac{0}{1*1^2} = 0 \quad \therefore \lim_{R\to\infty}\int_{z=-R,\,C_2}^{R}\frac{z^2dz}{(z^2+9)(z^2+4)^2}=0$$

$$\therefore I = \frac12\left[\oint_{C_3}+\oint_{C_4}-0\right] = \frac12\left(\frac{26\pi}{200}-\frac{24\pi}{200}\right) = \frac12\cdot\frac{2\pi}{200} = \frac{\pi}{200}, \quad \therefore \text{OK}$$

$\frac{6}{186}$

Let $\int_{0}^{\infty} \frac{x^2 dx}{x^6+1} = I$



$\therefore\ 2I = \int_{-\infty}^{\infty} \frac{x^2 dx}{x^6+1} = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2 dx}{x^6+1} =$

$= \lim_{R\to\infty} \int_{C_1} \frac{z^2 dz}{z^6+1} = \lim_{R\to\infty} \left(\oint_{C_1+C_2} - \int_{C_2}\right) \frac{z^2 dz}{z^6+1}$

Singularities at $z^6+1=0$

$\therefore\ z^6 = -1 = 1\angle \pi+2k\pi$

$\therefore\ z = 1\angle (2k+1)\pi/6 = \pm i,\ \frac{\pm\sqrt{3}\pm i}{2}$

$\therefore$ The only singularities within C_1+C_2 are i, $\frac{\pm\sqrt{3}+i}{2}$ (three)

$\therefore\ \oint_{C_1+C_2} \frac{z^2 dz}{z^6+1} = \oint_{C_3: |z-i|=\epsilon} \frac{z^2 dz}{z^6+1} + \oint_{C_4: \left|z-(\sqrt{3}+i)/2\right|=\epsilon} z^2dz/(z^6+1) + \oint_{C_5: \left|z-(-\sqrt{3}+i)/2\right|=\epsilon} z^2dz/(z^6+1)$

$\oint_{C_3} \frac{z^2 dz}{z^6+1} = 2\pi i \operatorname{Res} \frac{z^2}{z^6+1}\Big|_i = 2\pi i \left(\frac{(z-i)z^2}{z^6+1}\right)\Big|_i = 2\pi i\left(\frac{z^2}{6z^5}\right)\Big|_i = 2\pi i\left(\frac{-1}{6i}\right) = -\pi/3$

$+\ \oint_{C_4} \frac{z^2 dz}{z^6+1} = 2\pi i \operatorname{Res} \frac{z^2}{z^6+1}\Big|_{\frac{\sqrt{3}+i}{2}} = 2\pi i\left(\frac{(z-(\sqrt{3}+i)/2)z^2}{z^6+1}\right)\Big|_{\frac{\sqrt{3}+i}{2}} = 2\pi i\left(\frac{z^2}{6z^5}\right)\Big|_{\frac{\sqrt{3}+i}{2}}$

$= 2\pi i/6z^3\Big|_{1\angle 30^\circ} = \frac{\pi i}{3\angle 90^\circ} = \pi i/3i = \pi/3$

$+\ \oint_{C_5} \frac{z^2 dz}{z^6+1} = 2\pi i \operatorname{Res} \frac{z^2}{z^6+1}\Big|_{\frac{-\sqrt{3}+i}{2}} = 2\pi i\left(\frac{(z-(-\sqrt{3}+i)/2)z^2}{z^6+1}\right)\Big|_{\frac{-\sqrt{3}+i}{2}} = 2\pi i\left(\frac{z^2}{6z^5}\right)\Big|_{\frac{-\sqrt{3}+i}{2}}$

$= 2\pi i/6z^3\Big|_{1\angle 150^\circ} = \frac{\pi i}{3\angle 450^\circ} = \frac{\pi i}{3\angle 90^\circ} = \pi i/3i = \pi/3$

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$\therefore\ \oint_{C_1+C_2} z^2dz/(z^6+1) = \left(\oint_{C_3} + \oint_{C_4} + \oint_{C_5}\right)\frac{z^2 dz}{z^6+1} = -\pi/3+\pi/3+\pi/3 = \pi/3$

$$f \quad \left| \int_{C_2 : |z|=R,\ \theta\in[0,\pi]} z^2 dz/(z^6+1) \right| \le \int_{C_2} \left|\frac{z^2}{z^6+1}\right| |dz| \le \int_{C_2} \frac{|z|^2}{|z|^6-1} |dz| =$$

$$= \int_{C_2} \frac{R^2}{R^6-1} |dz| = \frac{R^2}{R^6-1} \int_{C_2} |dz| = \frac{R^2}{R^6-1} \cdot \pi R = \frac{\pi R^3}{R^6-1} \to 0 \quad \text{as } R\to\infty$$

$$\therefore \lim_{R\to\infty} \int_{C_2} z^2 dz/(z^6+1) = 0$$

$$\therefore \quad 2I = \lim_{R\to\infty} \left(\oint_{C_1+C_2} - \int_{C_2} \right) \frac{z^2 dz}{z^6+1} = \lim_{R\to\infty} \pi/3 - \lim_{R\to\infty} \int_{C_2} \frac{z^2 dz}{z^6+1} =$$

$$= \frac{\pi}{3} - 0 = \pi/3$$

$$\therefore \quad 2I = \pi/3 \qquad \therefore \quad I = \pi/6$$

$$\therefore \quad \int_0^\infty \frac{x^2 dx}{x^6+1} = \pi/6$$

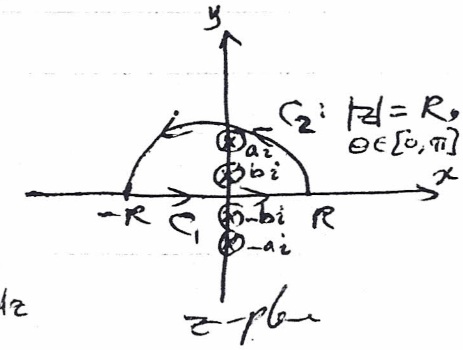
$\therefore$ OK

$\boxed{\dfrac{9}{186}}$

$$I = \int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} \qquad a>b>0$$

$$= \int_{-\infty}^{\infty} \frac{Re(e^{ix})\, dx}{(x^2+a^2)(x^2+b^2)} = Re \int_{-\infty}^{\infty} \frac{e^{ix} dx}{(x^2+a^2)(x^2+b^2)} = Re\, I'$$

$$\therefore I' = \int_{-\infty}^{\infty} \frac{e^{ix} dx}{(x^2+a^2)(x^2+b^2)} = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$$



where, $f(x) = \dfrac{e^{ix}}{(x^2+a^2)(x^2+b^2)}$

$$= \lim_{R\to\infty} \int_{\substack{z=-R\\ C_1}}^{z=R} f(z)\, dz = \lim_{R\to\infty} \left(\oint_{C_1+C_2} - \int_{\substack{z=-R\\ C_2}}^{R} \right) f(z) dz$$

$\because$ Singularities of $f(z)$ are at $z_0 = \pm ai, \pm bi$

$$\therefore I' = \lim_{R\to\infty} \left(\oint_{C_3} + \oint_{C_4} - \int_{\substack{z=-R\\ C_2}}^{R} \right) f(z) dz \quad, \text{ where } C_3: |z-ai| = \epsilon_1,\ C_4: |z-bi| = \epsilon_2$$

$$\therefore \oint_{C_3} = \oint_{C_3} \frac{e^{iz} dz}{(z^2+a^2)(z^2+b^2)} = \oint_{C_3} \frac{e^{iz}/[(z^2+b^2)(z+ai)]}{(z-ai)} dz = 2\pi i \left(\frac{e^{iz}}{(z^2+b^2)(z+ai)} \right)\Bigg|_{z=ai} =$$

$$= 2\pi i * \frac{e^{-a}}{(-a^2+b^2)2ai} = \frac{-\pi e^{-a}}{a(a^2-b^2)} = \frac{\pi}{a^2-b^2}\left(-\frac{e^{-a}}{a}\right)$$

$$\& \oint_{C_4} = \oint_{C_4} \frac{e^{iz} dz}{(z^2+a^2)(z^2+b^2)} = \oint_{C_4} \frac{e^{iz}/[(z^2+a^2)(z+bi)]}{(z-bi)} dz = 2\pi i \left(\frac{e^{iz}}{(z^2+a^2)(z+bi)} \right)\Bigg|_{z=bi} =$$

$$= 2\pi i * \frac{e^{-b}}{(-b^2+a^2)(2bi)} = \frac{\pi e^{-b}}{b(a^2-b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}\right)$$

$$\& \left| \int_{\substack{z=-R\\ C_2}}^{R} f(z) dz \right| \le \int_{\substack{z=-R\\ C_2}}^{R} \frac{|e^{iz}||dz|}{|(z^2+a^2)||(z^2+b^2)|} \le \int_{\substack{z=-R\\ C_2}}^{R} \frac{|e^{ix-y}||dz|}{||z^2|-|a^2||\cdot||z^2|-|b^2||} =$$

$$= \frac{\int_{z=-R, C_2}^{R} |e^{ix}\cdot e^{-y}||dz|}{(R^2-a^2)(R^2-b^2)} = \frac{\int_{z=-R, C_2}^{R} e^{-y}\cdot |dz|}{(R^2-a^2)(R^2-b^2)} \le \frac{e^{0}\, \pi R}{(R^2-a^2)(R^2-b^2)}$$ (Note: in such problems you can also use $|z| = R$, $\theta \in [-\pi, 0]$ because, though you will have e^{y} but $y<0$ and e^{y} is bounded by $e^0 = 1$)

$$= \frac{\pi R}{(R^2-a^2)(R^2-b^2)} = \text{as } R\to\infty \ \frac{\pi/R^3}{(1-a^2/R^2)(1-b^2/R^2)} = \frac{0}{1\cdot 1} = 0$$

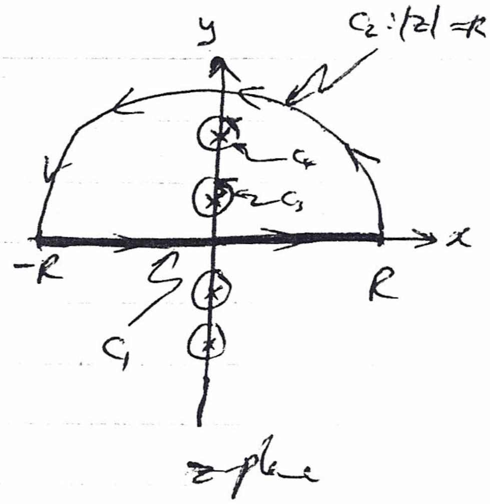
$$\therefore I' = \frac{\pi}{a^2-b^2}\left(-\frac{e^{-a}}{a} + \frac{e^{-b}}{b}\right) - 0 = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) = I, \quad \therefore \text{OK}$$

$a>b>0$

$\boxed{\dfrac{14}{186}}$

$$I=\int_{-\infty}^{\infty}\frac{x\sin x\,dx}{(x^2+1)(x^2+4)}=$$

$$=\lim_{R\to\infty}\int_{-R}^{R}\frac{x\,\mathrm{Im}\,e^{ix}}{(x^2+1)(x^2+4)}\,dx=$$



$$=\mathrm{Im}\lim_{R\to\infty}\int_{-R}^{R}\frac{xe^{ix}dx}{(x^2+1)(x^2+4)}=$$

$$=\mathrm{Im}\lim_{R\to\infty}\int_{C_1}\underbrace{\frac{ze^{iz}dz}{(z^2+1)(z^2+4)}}_{f(z)\,dz}=$$

$$=\mathrm{Im}\lim_{R\to\infty}\int_{C_1+C_2-C_2}f(z)\,dz=$$

$$=\mathrm{Im}\lim_{R\to\infty}\Big(\oint_{C_1+C_2}-\int_{C_2}\Big)f(z)\,dz=\mathrm{Im}\lim_{R\to\infty}\Big(\oint_{\substack{C_3\\|z-i|=\epsilon}}+\oint_{\substack{C_4\\|z-2i|=\epsilon}}-\int_{C_2}\Big)f(z)\,dz$$

$$\oint_{C_3:\,|z-i|=\epsilon}f(z)\,dz=\oint\frac{ze^{iz}/[(z^2+4)(z+i)]}{z-i}\,dz=2\pi i\left.\frac{ze^{iz}}{(z^2+4)(z+i)}\right|_{i}=$$

$$=2\pi i\cdot\frac{i\cdot e^{-1}}{(-1+4)(2i)}=\frac{\pi i e^{-1}}{3}=\frac{\pi i}{3e}$$

$$\oint_{C_4:\,|z-2i|=\epsilon}f(z)\,dz=\oint\frac{ze^{iz}/[(z^2+1)(z+2i)]}{z-2i}\,dz=2\pi i\left.\frac{ze^{iz}}{(z^2+1)(z+2i)}\right|_{2i}=$$

$$=2\pi i\cdot\frac{2i\cdot e^{-2}}{(-4+1)(4i)}=\frac{\pi i e^{-2}}{-3}=-\frac{\pi i}{3e^2}$$

$$\left|\int_{C_2}f(z)\,dz\right|\le\int\left|\frac{ze^{iz}dz}{(z^2+1)(z^2+4)}\right|\le\int\frac{|z|\,|dz|}{(|z|^2-1)(|z|^2-4)}=\int\frac{R\,|dz|}{(R^2-1)(R^2-4)}=$$

$$=\int\frac{|dz|/R^3}{(1-1/R^2)(1-4/R^2)}=\frac{\pi R/R^3}{(1-1/R^2)(1-4/R^2)}=(\text{as }R\to\infty)\ \frac{\pi/R^2}{1\times1}=0$$

$$\therefore\ I=\mathrm{Im}\lim_{R\to\infty}\Big(\oint_{C_3}+\oint_{C_4}-\int_{C_2}\Big)f(z)\,dz=\mathrm{Im}\lim_{R\to\infty}\frac{\pi i}{3e}-\frac{\pi i}{3e^2}-0=$$

$$=\frac{\pi}{3e}-\frac{\pi}{3e^2}=\frac{\pi}{3e^2}(e-1).$$

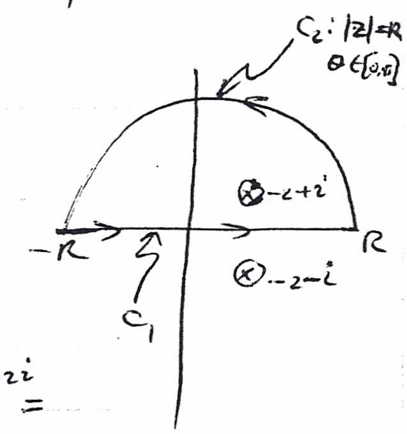
$\boxed{\frac{15}{186}}$

$$I=\int_{-\infty}^{\infty}\frac{\sin x\,dx}{x^2+4x+5} = \lim_{R\to\infty}\int_{-R}^{R}\frac{\operatorname{Im} e^{ix}}{x^2+4x+5}\cdot dx = \operatorname{Im}\left(\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx\right) = \operatorname{Im} I'$$

where, $f(x)=\frac{e^{ix}}{x^2+4x+5}$, $\therefore f(z)=\frac{e^{iz}}{z^2+4z+5}$, singularity at $z_0=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$

$$\therefore I'=\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty}\int_{z=-R,\ C_1}^{R} f(z)\,dz = \lim_{R\to\infty}\left(\oint_{C_1+C_2}-\int_{C_2}\right)f(z)\,dz=$$

$$=\lim_{R\to\infty}\left(\oint_{C_3}-\int_{C_2}\right)f(z)\,dz, \text{ where } C_3: |z-(-2+i)|=\epsilon$$



$$\therefore \oint_{C_3} = (I' \text{ Cauchy principal value}) = \oint_{C_3}\frac{e^{iz}}{(z^2+4z+5)}\,dz=$$

$$=\oint_{C_3}\frac{e^{iz}/(z+2+i)}{(z+2-i)}\,dz = 2\pi i\left(\left.\frac{e^{iz}}{z+2+i}\right|_{z=-2+i}\right)=$$

$$=2\pi i * \frac{e^{i(-2+i)}}{-2+i+2+i} = 2\pi i*\frac{e^{-2i-1}}{2i} = \pi e^{-1}\cdot e^{-2i}=$$

$=\frac{\pi}{e}(\cos 2 - i\sin 2)$ $\therefore$ CPV (Cauchy principal value) of $I'=\frac{\pi}{e}(\cos 2 - i\sin 2)$

$\therefore$ CPV of $I = \operatorname{Im}(\text{CPV of } I') = \operatorname{Im}\left(\frac{\pi}{e}(\cos 2 - i\sin 2)\right) = -\frac{\pi}{e}\sin 2$

(Note:

$$\left|\int_{z=-R,\ C_2}^{R} f(z)\,dz\right| = \left|\int_{z=-R,\ C_2}^{R}\frac{e^{iz}}{z^2+4z+5}\,dz\right| \le \int_{z=-R,\ C_2}^{R}\frac{|e^{ix-y}|\,|dz|}{|z+2+i|\,|z+2-i|}=$$

$$=\int_{z=-R,\ C_2}^{R}\frac{|e^{ix}|\cdot|e^{-y}|\,|dz|}{|z+2+i|\,|z+2-i|} \le \int_{z=-R,\ C_2}^{R}\frac{|e^{-y}|\,|dz|}{\big||z|-|2+i|\big|\cdot\big||z|-|2-i|\big|}=$$

$$=\int_{z=-R,\ C_2}^{R}\frac{e^{-y}|dz|}{(R-\sqrt5)(R-\sqrt5)} \le \frac{1*\pi R}{(R-\sqrt5)^2} = \frac{\pi/R}{(1-\sqrt5/R)^2} = \text{ as } R\to\infty\ \frac{0}{1^2}=0$$

$$\therefore I'=\frac{\pi}{e}(\cos 2 - i\sin 2) - 0 = \frac{\pi}{e}(\cos 2 - i\sin 2)$$

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$\therefore I=\operatorname{Im} I' = -\frac{\pi}{e}\sin 2$ = CPV of I = Im(CPV of I'), this is why they call it principal value because the rest reaches 0 as $R\to\infty$).

$\frac{1a}{19i}$) $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \oint_{C:\,|z|=1} \frac{(dz)/(iz)}{5+4\left(\frac{z^2-1}{2iz}\right)} =$

Let $|z|=1 \quad \therefore z = e^{i\theta}$

$\therefore \left(z-\frac{1}{z}\right) = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\therefore \sin\theta = (z^2-1)/(2iz)$

$\&\ dz = z\,i\,d\theta$

$= \oint_C \frac{dz}{5iz+2z^2-2} = I$

$\therefore 5iz+2z^2-2 = 0$ at $z = \frac{-5i \pm \sqrt{-25-4(2)(-2)}}{4}$

$= \frac{-5i \pm \sqrt{-9}}{4} = \frac{-5i \pm 3i}{4} = -2i$ or $-i/2$ and only $z = \frac{-i}{2}$ is in C

$\therefore I = \oint_C \frac{(dz)/[2(z+2i)]}{(z+i/2)} = 2\pi i \cdot \frac{1}{2(z+2i)}\Big|_{-i/2} = 2\pi i \frac{1}{2\left(-\frac{i}{2}+\frac{4i}{2}\right)} = \pi i \cdot \frac{1}{3i/2} = \frac{2\pi}{3}$

$\therefore \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi/3$

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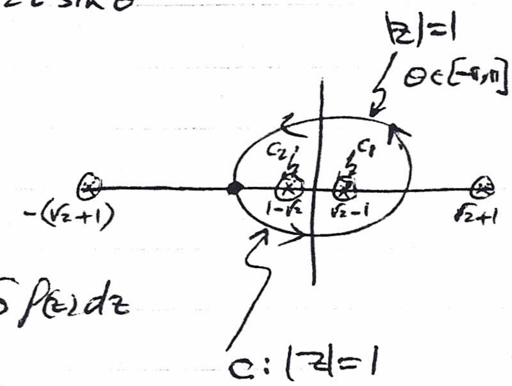
$\boxed{\frac{16}{191}}$

$$I=\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}\quad,\ \text{Let } z=e^{i\theta}\quad\therefore\ dz=e^{i\theta}\cdot i\,d\theta=z\,i\,d\theta$$

$$\& \quad z-\frac{1}{z}=e^{i\theta}-e^{-i\theta}=2i\sin\theta$$

$$\therefore\ I=\oint_C\frac{(dz)/iz}{1+[(z-\frac{1}{z})/2i]^2}=\oint_C\frac{4izdz}{4i^2z^2+(z^2-1)^2}=$$



$$=\oint_C\frac{4izdz}{-4z^2+z^4-2z^2+1}=\oint_C\frac{4izdz}{z^4-6z^2+1}=\oint_C f(z)dz$$

$$\therefore\ f(z)=\frac{4iz}{z^4-6z^2+1}\quad,\ \text{singularity at } z_0^2=\frac{6\pm\sqrt{36-4}}{2}=\frac{6\pm\sqrt{32}}{2}=3\pm2\sqrt{2}$$

$$\text{or}\quad z_0=\pm\sqrt{3\pm2\sqrt{2}}\simeq\pm\left\langle\begin{matrix}2.414\\0.414\end{matrix}\right.=\pm(\sqrt{2}\pm1)$$

$$\left(\text{Note: } 3\pm2\sqrt{2}=2+1\pm2\sqrt{2}=(\sqrt{2})^2\pm2\sqrt{2}+1=(\sqrt{2}\pm1)^2\right).$$

$$\therefore\ I=\oint_{C_1}f(z)dz+\oint_{C_2}f(z)dz\quad,\ \text{where } C_1: |z+1-\sqrt{2}|=\epsilon_1,\ C_2: |z+\sqrt{2}-1|=\epsilon_2$$

$$=2\pi i\{B_1+B_2\}\ ,\text{where } B_1, B_2 \text{ are residues of } f(z) \text{ at } (\sqrt{2}-1), (1-\sqrt{2}) \text{ respectively.}$$

$$\therefore\ B_1=\left(\frac{4iz(z+1-\sqrt{2})}{z^4-6z^2+1}\right)\Bigg|_{z=\sqrt{2}-1}=\frac{4iz}{(z-\sqrt{2}-1)(z+\sqrt{2}+1)(z+\sqrt{2}-1)}\Bigg|_{\sqrt{2}-1}=\frac{4i\cancel{(\sqrt{2}-1)}}{(-2)(2\sqrt{2})2\cancel{(\sqrt{2}-1)}}$$

$$=\frac{4i}{-8\sqrt{2}}=-\frac{i}{2\sqrt{2}}$$

$$\&\quad B_2=\left(\frac{4iz(z+\sqrt{2}-1)}{z^4-6z^2+1}\right)\Bigg|_{z=1-\sqrt{2}}=\frac{4iz}{(z-\sqrt{2}-1)(z+\sqrt{2}+1)(z+1-\sqrt{2})}\Bigg|_{1-\sqrt{2}}=\frac{4i\cancel{(1-\sqrt{2})}}{(-2\sqrt{2})(2)2\cancel{(1-\sqrt{2})}}=$$

$$=\frac{4i}{-8\sqrt{2}}=-\frac{i}{2\sqrt{2}}$$

$$\therefore\ I=2\pi i(B_1+B_2)=2\pi i\left(\frac{-i}{2\sqrt{2}}-\frac{i}{2\sqrt{2}}\right)=2\pi i\left(\frac{-\cancel{2}i}{\cancel{2}\sqrt{2}}\right)=2\pi i\left(\frac{-i}{\sqrt{2}}\right)=\frac{2\pi}{\sqrt{2}}$$

$$\therefore\ \int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\frac{2\pi}{\sqrt{2}}=\sqrt{2}\,\pi,\qquad \therefore \text{OK}$$

$\frac{2}{191}$

Let $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = I \qquad a \in (-1,1)$

Let C be $|z|=1$, $\theta \in [0, 2\pi]$ in the positive sense

$\therefore\ z = e^{i\theta} \quad \therefore\ dz = z\,i\,d\theta \quad \& \ \cos\theta = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$

$\therefore\ I = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \oint_C \frac{dz/iz}{1+a\left(\frac{z^2+1}{2z}\right)} = \frac{1}{i}\oint_C \frac{2\,dz}{2z+az^2+a} =$

$= \frac{2}{ai}\oint_C \frac{dz}{z^2+\frac{2z}{a}+1}$, singularities at $z = \frac{-2 \pm \sqrt{4-4a^2}}{2a} =$

$= (-1 \pm \sqrt{1-a^2})/a \quad \because\ a \in (-1,1) \quad \therefore$ Singularities are real and the only one lying within C is $(-1+\sqrt{1-a^2})/a$ (goes to 0 at $a=0$)

$\therefore\ I = \frac{2\cdot 2\pi i}{ai}\operatorname{Res}\left(\frac{1}{z^2+\frac{2z}{a}+1}\right)\Big|_{\frac{-1+\sqrt{1-a^2}}{a}} = (4\pi/a)\left(\frac{z-\left(\frac{-1+\sqrt{1-a^2}}{a}\right)}{z^2+\frac{2z}{a}+1}\right)\Big|_{\frac{-1+\sqrt{1-a^2}}{a}} = (4\pi)\left(\frac{1}{2az+2}\right)\Big|_{(-1+\sqrt{1-a^2})/a} = \frac{4\pi}{2\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}.$

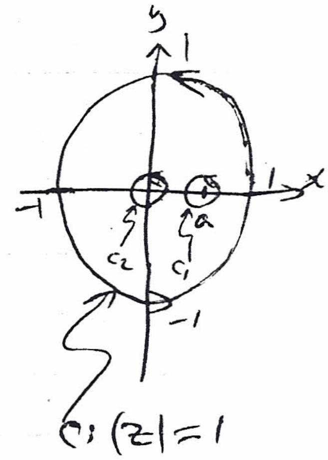
$\therefore$ OK

$\boxed{\frac{4}{191}}$

$$I=\int_0^{\pi}\frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2}\quad , \quad a\in(-1,1)$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2}$$. Let $|z|=1$ $\therefore z=e^{i\theta}$

$\therefore$ $dz = zi\,d\theta$ & $z+\frac{1}{z}=e^{i\theta}+e^{-i\theta}=2\cos\theta$ $\therefore \cos\theta=\frac{z^2+1}{2z}$



$$\therefore I=\frac{1}{2}\oint_{C:|z|=1}\frac{(2\cos^2\theta-1)\,d\theta}{1-2a\cos\theta+a^2}=$$

$$=\frac{1}{2}\oint_C\frac{\left[2\left(\frac{z^2+1}{2z}\right)^2-1\right]\frac{dz}{iz}}{1-\cancel{2}a\left(\frac{z^2+1}{\cancel{2}z}\right)+a^2}=$$

$$=\frac{1}{2i}\oint_C\frac{\frac{(z^2+1)^2-2z^2}{2z^2}}{(z-a(z^2+1)+a^2z)}\,dz=$$

$$=\frac{1}{-4ia}\oint_C\frac{z^4+\cancel{2z^2}+1-\cancel{2z^2}}{z^2\left(z^2-\left(\frac{1}{a}\right)(1+a^2)z+1\right)}\,dz\quad , \quad a\neq 0$$

Singularities at $z_0=\frac{\frac{1}{a}(1+a^2)\pm\sqrt{\left(\frac{1}{a}\right)^2(1+a^2)^2-4}}{2}=\frac{1+a^2\pm\sqrt{(1+a^2)^2-4a^2}}{2a}$

$$=\frac{1+a^2\pm\sqrt{1+2a^2+a^4-4a^2}}{2a}=\frac{1+a^2\pm\sqrt{1-2a^2+a^4}}{2a}=$$

$$=\frac{1+a^2\pm(1-a^2)}{2a}=\begin{cases} + : & \frac{1}{a} \text{ outside } |z|=1 \text{ since } a\in(-1,1)\\ - : & a \text{ inside } |z|=1 \text{ since } a\in(-1,1)\end{cases}$$

$$\therefore I=\frac{1}{-4ai}\oint_C\underbrace{\frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})}}_{f(z)\,dz}\,dz=\frac{i}{4a}\left(\oint_{C_1}+\oint_{C_2}\right)f(z)\,dz$$

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$$\therefore \oint_{C_1:|z-a|=\epsilon}f(z)\,dz=2\pi i\left[\frac{(z^4+1)}{z^2(z-\frac{1}{a})}\right]\Bigg|_a=2\pi i * \frac{a^4+1}{a^2(a-\frac{1}{a})}=\frac{a^4+1}{a(a^2-1)}\cdot 2\pi i$$

$$\& \oint_{C_2:|z|=\epsilon}f(z)\,dz=2\pi i\left[\frac{z^4+1}{(z-a)(z-\frac{1}{a})}\right]'\Bigg|_0=2\pi i\cdot\frac{4z^3(z-a)(z-\frac{1}{a})-(z^4+1)(2z-a-\frac{1}{a})}{(z-a)^2(z-\frac{1}{a})^2}\Bigg|_0=2\pi i\cdot\frac{a^2+1}{a}$$

$$\therefore I=\frac{i}{4a}\cdot\left[\oint_{C_1}+\oint_{C_2}\right]f(z)\,dz=\frac{i}{4a}\left[\frac{a^4+1}{a(a^2-1)}\cdot 2\pi i+\frac{a^2+1}{a}\cdot 2\pi i\right]=\frac{2\pi i^2}{4a}\cdot\frac{a^4+1+(a^2+1)(a^2-1)}{a(a^2-1)}=$$

$$=\frac{-\pi}{2a}\cdot\frac{a^4+\cancel{1}+a^4-\cancel{1}}{a(a^2-1)}=\frac{-\pi(2a^4)}{2a^2(a^2-1)}=-\frac{\pi a^2}{a^2-1}=\frac{\pi a^2}{1-a^2}$$, $\therefore$ OK.

$\boxed{\frac{5}{191}}$ Let $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = I$, $a \in (1,\infty)$

$\therefore 2I = 2\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}$

Let $z = e^{i\theta}$ $\therefore |z| = 1$ $\therefore$ $\theta \in [0, 2\pi]$ describes circle $|z| = 1$ ccw

$\therefore$ $dz = zi\,d\theta$ & $\cos\theta = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z}$

$\therefore$ $2I = \oint_{C:\,|z|=1} \frac{dz/iz}{(a+(z^2+1)/2z)^2} = \oint_C \frac{dz}{iz\left(\frac{2az+z^2+1}{2z}\right)^2} = \frac{4}{i}\oint_C \frac{z\,dz}{(z^2+2az+1)^2}$

$\therefore$ Singularities at $z = \frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1}$

Only the singularity of $-a+\sqrt{a^2-1}$ is within C for $a \in (1,\infty)$

$\therefore$ $2I = \frac{4}{i}\oint_C \frac{z\,dz}{(z^2+2az+1)^2} = \frac{4}{i}\oint_{C':\,|z+a-\sqrt{a^2-1}|=\epsilon} z\,dz/(z^2+2az+1)^2 =$

$= \frac{4}{i} * 2\pi i \operatorname{Res}\left(\frac{z}{(z^2+2az+1)^2}\right)\Bigg|_{-a+\sqrt{a^2-1}} = 8\pi\left(\frac{(z+a-\sqrt{a^2-1})^2 z}{(z^2+2az+1)^2}\right)'\Bigg|_{-a+\sqrt{a^2-1}} =$

$= 8\pi\left(\frac{z}{(z+a+\sqrt{a^2-1})^2}\right)'\Bigg|_{-a+\sqrt{a^2-1}} = 8\pi\left[\frac{(z+a+\sqrt{a^2-1})^2 - 2z(z+a+\sqrt{a^2-1})}{(z+a+\sqrt{a^2-1})^4}\right]_{-a+\sqrt{a^2-1}} =$

$= 8\pi\left[\frac{-(-a+\sqrt{a^2-1})+a+\sqrt{a^2-1}}{(-a+\sqrt{a^2-1}+a+\sqrt{a^2-1})^3}\right] = 8\pi * \frac{2a}{(2\sqrt{a^2-1})^3} = \frac{16\pi a}{8(a^2-1)^{3/2}}$

$\therefore$ $2I = \frac{2\pi a}{(a^2-1)^{3/2}}$ $\therefore$ $I = \frac{\pi a}{(a^2-1)^{3/2}}$

$\therefore$ $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \pi a/(a^2-1)^{3/2}$

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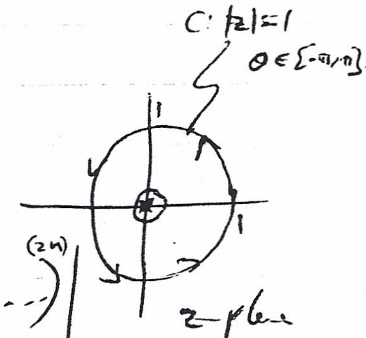
$I = \int_0^{\pi} \sin^{2n}\theta \, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta \, d\theta$, because $\sin^{2n}\theta$ is even function

Let $z = e^{i\theta}$ ∴ $dz = iz\,d\theta$, $z - \frac{1}{z} = 2i\sin\theta$ ∴ $\sin\theta = \frac{z^2-1}{2iz}$

$$\therefore I = \frac{1}{2}\oint_{C\,(-\pi)}^{\pi} \left(\frac{z^2-1}{2iz}\right)^{2n}\frac{dz}{iz} = \frac{1}{2}\oint_C \frac{(z^2-1)^{2n}dz}{(2iz)^{2n}\cdot iz} =$$

$$= \frac{1}{2}\oint_C \frac{(z^2-1)^{2n}dz}{2^{2n}\cdot i^{2n+1}\cdot z^{2n+1}} = \frac{1}{2^{2n+1}\cdot i^{2n+1}}\oint_C \frac{(z^2-1)^{2n}dz}{z^{2n+1}}$$



$$= \frac{1}{(2i)^{2n+1}} * 2\pi i \left.\frac{\left[(z^2-1)^{2n}\right]^{(2n)}}{(2n)!}\right|_{z=0} =$$

$$= \frac{2\pi i}{(2i)^{2n+1}} * \left.\frac{\left(z^{4n} - 2n z^{4n-2} + \cdots + \binom{2n}{k} z^{4n-2k}\cdot(-1)^k + \cdots\right)^{(2n)}}{(2n)!}\right|_0$$

$$= \frac{\pi}{(2i)^{2n}(2n)!} * \left.\left(\sum_{k=0}^{2n}\binom{2n}{k} z^{4n-2k}(-1)^k\right)^{(2n)}\right|_0 =$$

$$= \frac{\pi}{(2i)^{2n}\cdot(2n)!} * \sum_{k=0}^{2n}\left.\left(\frac{(2n)! * (-1)^k z^{4n-2k}}{k!\,(2n-k)!}\right)^{(2n)}\right|_0 =$$

$$= \frac{\pi\,\cancel{(2n)!}}{(2i)^{2n}\cdot\cancel{(2n)!}} * \sum_{k=0}^{2n}\left.\left(\frac{(-1)^k z^{4n-2k}}{k!\,(2n-k)!}\right)^{(2n)}\right|_0 =$$

$$= \frac{\pi}{(2i)^{2n}} * \sum_{k=0}^{n\ \text{(rest vanish)}} \left.\frac{(-1)^k (4n-2k)(4n-2k-1)\cdots(4n-2k-(2n-1))\,z^{2n-2k}}{k!\,(2n-k)!}\right|_{z=0}$$

$$= \frac{\pi}{(2i)^{2n}} * \sum_{k=0}^{n}\left.\frac{(-1)^k(4n-2k)(\cdots\cdots)(2n-2k+1)\,z^{2n-2k}}{k!\,(2n-k)!}\right|_0 =$$

Note: $(a+b)^m = a^m + m a^{m-1}b + \cdots + \binom{m}{k}a^{m-k}\cdot b^k + \cdots = \sum_{k=0}^{m}\binom{m}{k}a^{m-k}\cdot b^k$

$\binom{m}{k} = \frac{m!}{k!\,(m-k)!}$

put $a = z^2$, $b = -1$, $m = 2n$.

$$= \frac{\pi}{(2i)^{2n}} * \frac{(-1)^n(4n-2n)(\cdots)(1)*1}{(n!)(2n-n)!} \text{ (rest terms vanish at } z=0) =$$

$$= \frac{\pi}{(2i)^{2n}} * \frac{(i^2)^n(2n)\cdots(1)}{(n!)^2} = \pi\left(\frac{i}{2i}\right)^{2n}\cdot\frac{(2n)!}{(n!)^2} = \frac{\pi}{2^{2n}}\cdot\frac{(2n)!}{(n!)^2}$$

$$\therefore I = \int_0^{\pi}\sin^{2n}\theta\,d\theta = \frac{(2n)!\,\pi}{2^{2n} * (n!)^2}, \quad \therefore \text{OK.}$$

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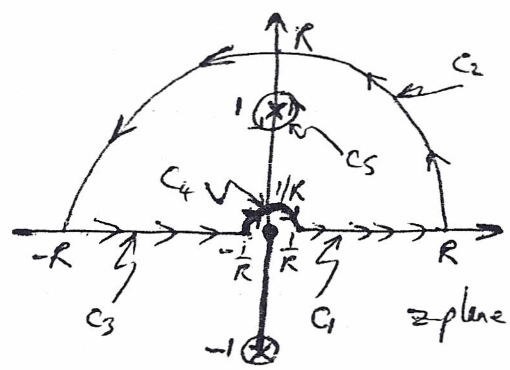
$\boxed{\frac{8a}{192}}$ $\quad I = \int_0^\infty \frac{\mathrm{Log}\, x \, dx}{1+x^2} =$



$$= \lim_{R\to\infty} \int_{1/R}^{R} \frac{\mathrm{Log}\, x\, dx}{1+x^2} =$$

$$= \lim_{R\to\infty} \int_{C_1} f(z)\, dz \quad , \text{where}$$

$f(z) = \dfrac{\log z}{1+z^2}$, $\log z$ defined on the branch $|z|>0$, $\angle z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$\therefore$ $f(z)$ has singularities at $0, \pm i$

Consider the contour $C_1 + C_2 + C_3 + C_4$, where:

C_1: $y=0$, $x \in \left[\frac{1}{R}, R\right]$,

C_2: $|z| = R$, $\theta \in [0, \pi]$,

C_3: $y = 0$, $x \in [-R, -1/R]$ &

C_4: $|z| = \frac{1}{R}$, $\theta \in [\pi, 0]$.

This contour is closed and avoids the singularity and cut of the $\log x$, and enclose the singularity of $+i$

$$\therefore I = \lim_{R\to\infty} \int_{C_1} f(z)dz = \lim_{R\to\infty} \left(\oint_{C_1+C_2+C_3+C_4} - \int_{C_2} - \int_{C_3} - \int_{C_4} \right) f(z)\, dz$$ III

$$\therefore \oint_{C_1+C_2+C_3+C_4} f(z)dz = \oint_{C_5: |z-i| = \epsilon} f(z)\, dz = \oint_{C_5} \frac{\log z\, dz}{(z+i)(z-i)} = 2\pi i \left. \frac{\log z}{z+i} \right|_i = 2\pi i \frac{\log i}{2i} =$$
$$= \pi \log(1 \angle \pi/2) =$$
$$= \pi(\ln 1 + i\pi/2) = \frac{i\pi^2}{2}$$

$$\left|\int_{C_2} f(z)\,dz\right| \le \int_{C_2} |f(z)|\,|dz| = \int_{C_2} \frac{|\mathrm{Log}\, z|}{|1+z^2|}\cdot|dz| \le \int_{C_2} \frac{|\ln|z| + i\angle z|\,|dz|}{|1-|z|^2|}$$

$$\le \int \frac{|\ln R + i\pi|\,|dz|}{|1-R^2|} < \frac{(\ln R)+\pi}{R^2-1}\cdot\int|dz| = \frac{\pi+\ln R}{R^2-1}\cdot\pi R \xrightarrow[\text{as } R\to\infty]{} 0$$

$$\therefore \lim_{R\to\infty}\int_{C_2} f(z)\,dz = 0$$

$$\left|\int_{C_4} f(z)\,dz\right| \le \int_{C_4} |f(z)|\,|dz| = \int_{C_4} \frac{|\mathrm{Log}\, z|\,|dz|}{|1+z^2|} \le \int_{C_4} \frac{|\ln|z| + i\angle z|\,|dz|}{|1-|z|^2|} \le$$

$$\le \frac{|\ln(\frac{1}{R}) + i\pi|}{|1-(\frac{1}{R})^2|}\int|dz| < \frac{|-\ln R| + \pi}{(R^2-1)/R^2}\cdot\frac{\pi}{R} = \frac{\pi+\ln R}{R^2-1}\cdot\pi R \xrightarrow[\text{as } R\to\infty]{} 0$$

$$\therefore \lim_{R\to\infty}\int_{C_4} f(z)\,dz = 0$$

$$\int_{C_3} f(z)\,dz = \int_{C_3} \frac{\mathrm{Log}\, z\,dz}{1+z^2} = \int_{C_3} \frac{(\ln|z| + i\pi)\,dz}{1+z^2} = \int_{C_3} \frac{\ln|z|\,dz}{1+z^2} + \int_{C_3} \frac{i\pi\,dz}{1+z^2}$$

$$= \int_{-R}^{-1/R} \frac{\ln|x|\,dx}{1+x^2} + i\pi\int_{-R}^{-1/R} \frac{dx}{1+x^2} = \int_{-R}^{-1/R} \frac{\ln(-x)\,dx}{1+x^2} + i\pi \tan^{-1}x\Big|_{-R}^{-1/R} =$$

x is negative $\qquad$ x is negative

$$= \int_{x'=-x=R}^{1/R} \frac{-\ln x'\,dx'}{1+x'^2} + i\pi\left[\tan^{-1}\left(\frac{-1}{R}\right) - \tan^{-1}(-R)\right] = \int_{1/R}^{R} \frac{\ln x'\,dx'}{1+x'^2} + i\pi\left(\tan^{-1}R - \tan^{-1}\frac{1}{R}\right)$$

$$= \int_{1/R}^{R} \frac{\mathrm{Log}\, x\,dx}{1+x^2} + i\pi\tan^{-1}\frac{R-\frac{1}{R}}{1+R\cdot\frac{1}{R}} = \int_{1/R}^{R} \frac{\mathrm{Log}\, x\,dx}{1+x^2} + i\pi\tan^{-1}\left(\frac{R^2-1}{2R}\right)$$

$$\therefore\ I = \lim_{R\to\infty}\left(\oint_{C_1+C_2+C_3+C_4} - \int_{C_2} - \int_{C_3} - \int_{C_4}\right) f(z)\,dz = \lim_{R\to\infty}\oint_{C_1+C_2+C_3+C_4} f(z)\,dz - \lim_{R\to\infty}\int_{C_2} f(z)\,dz$$

$$-\lim_{R\to\infty}\int_{C_3} f(z)\,dz - \lim_{R\to\infty}\int_{C_4} f(z)\,dz = \lim_{R\to\infty}\frac{i\pi^2}{2} - 0 - \lim_{R\to\infty}\left(\int_{1/R}^{R}\frac{\mathrm{Log}\, x\,dx}{1+x^2} +\right.$$

$$\left.+ i\pi\tan^{-1}\left(\frac{R^2-1}{2R}\right)\right) - 0 = i\pi^2/2 - I - \lim_{R\to\infty} i\pi\tan^{-1}\left(\frac{R}{2}\right) = i\pi^2/2 - I - i\pi^2/2 = -I$$

$$\therefore\ I = -I \Rightarrow 2I = 0 \quad \therefore I = 0 \quad \therefore \int_0^\infty \frac{\mathrm{Log}\, x\,dx}{1+x^2} = 0 \quad \therefore \text{OK.}$$

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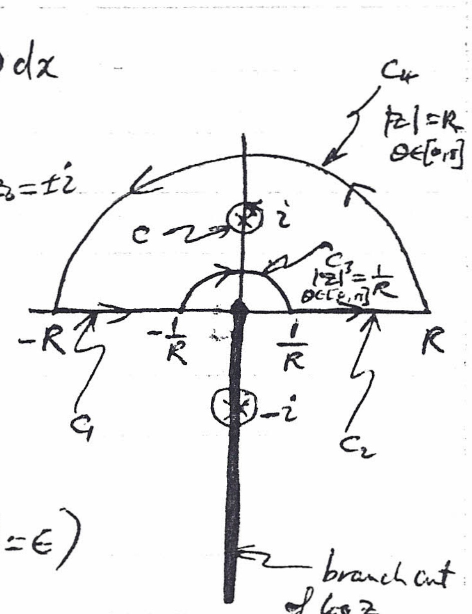
$\boxed{\frac{8b}{192}}$

$$I=\int_0^\infty \frac{\text{Log}\, x}{(x^2+1)^2}\,dx =$$

$$= \lim_{R\to\infty}\int_0^R \frac{\text{Log}\, x}{(x^2+1)^2}\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx$$

where $f(x)=\frac{\text{Log}\, x}{(x^2+1)^2}$ $\therefore$ $f(z)$ has singularities at $z_0=\pm i$



$$=\lim_{R\to\infty}\left[\int_{z=\frac{1}{R},\,C_2}^{R} f(z)\,dz\right]=$$

$$=\lim_{R\to\infty}\left[\left(\oint_{C_1+C_3+C_2+C_4} - \int_{z=-\frac{1}{R},\,C_3}^{z=\frac{1}{R}} - \int_{z=R,\,C_4}^{z=-R} - \int_{z=-R,\,C_1}^{-\frac{1}{R}}\right)f(z)\,dz\right]$$

$$\therefore \oint_{C_1+C_3+C_2+C_4} f(z)\,dz = \oint_C \frac{\text{Log}\, z}{(z^2+1)^2}\,dz = (C: |z-i|=\epsilon)$$

Log z defind on the branch $|z|>0,\ \theta\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$

$$=\oint_C \frac{(\log z)/(z+i)^2}{(z-i)^2}\,dz = 2\pi i\, B$$

$$B=\left(\frac{\log z}{(z+i)^2}\right)'\Bigg|_{z=i} = \frac{\frac{1}{z}(z+i)^2-2(z+i)\log z}{(z+i)^4}\Bigg|_{i} = \frac{\frac{(2i)^2}{i}-2(2i)\log i}{(2i)^4}=$$

$$=\frac{4i-4i(\ln 1^{0} + i\pi/2)}{16} = \frac{i}{4}(1-i\pi/2) = \frac{1}{8}(\pi+2i) \quad \therefore \oint_C f(z)\,dz = \frac{\pi}{4}(-2+i\pi)$$

f $$\left|\int_{C_3} f(z)\,dz\right| = \left|\int_{C_3}\frac{\log z}{(z^2+1)^2}\,dz\right| \le \underset{|z|=\frac{1}{R}}{\frac{|\ln(\frac{1}{R})+i\pi| * \pi(\frac{1}{R})}{\big||z^2|-|1|\big|^2}} \le \frac{[|-\ln R|+|i\pi|](\pi/R)}{((\frac{1}{R})^2-1)^2}=$$

$$=\frac{[(\ln R)+\pi]\,\pi/R}{((\frac{1}{R})^2-1)^2} = \frac{\pi\left(\frac{\ln R}{R}+\frac{\pi}{R}\right)}{((\frac{1}{R})^2-1)^2} = (\text{as } R\to\infty)\ \pi\frac{\ln R}{R} = \frac{\pi/R}{1}=0$$

f $$\left|\int_{C_4} f(z)\,dz\right| = \left|\int_{C_4}\frac{\log z}{(z^2+1)^2}\,dz\right| \le \frac{|\ln R + i\pi| * \pi R}{\big||z^2|-1\big|^2} \le \frac{((\ln R)+\pi)\pi R}{(R^2-1)^2}=$$

$$=\frac{\frac{\pi\ln R}{R^3}+\frac{\pi^2}{R^3}}{(1-1/R^2)^2} = (\text{as }R\to\infty)\ \frac{\pi\ln R}{R^3} \underset{|z|=R}{=} \frac{\pi/R}{3R^2} = \frac{\pi/3}{R^3}=0$$

f $$\int_{-R,\,C_1}^{-1/R} f(z)\,dz = \int_{-R,\,C_1}^{-1/R}\frac{\log z}{(z^2+1)^2}\,dz = \int_{-R,\,C_1}^{-1/R}\frac{\ln|x|+i\pi}{(x^2+1)^2}\,dx = \int_{x=-R,\,C_1}^{-1/R}\frac{\ln(-x)}{(x^2+1)^2}\,dx + i\pi\int_{x=-R,\,C_1}^{-1/R}\frac{dx}{(x^2+1)^2}=$$

x negative

$$=-\int_{x'=-x=R,\,C_1}^{1/R}\frac{\ln(-x)\,d(-x)}{((-x)^2+1)^2} - i\pi\int_{x'=-x=R,\,C_1}^{1/R}\frac{d(-x)}{((-x)^2+1)^2} = \int_{x'=1/R}^{R}\frac{\ln x'\,dx'}{(x'^2+1)^2} + i\pi\int_{x'=\frac{1}{R}}^{R}\frac{dx'}{(1+x'^2)^2}$$

x' real　x' real

$$\therefore I = \lim_{R\to\infty}\left[\oint_{C_1+C_3+C_2+C_4} - \int_{-1/R}^{-R}\!\!\!{}_{C_3} \to 0 - \int_{R}^{-R}\!\!\!{}_{C_4} \to 0 - \int_{-R}^{-1/R}\!\!\!{}_{C_1}\right] f(z)\,dz =$$

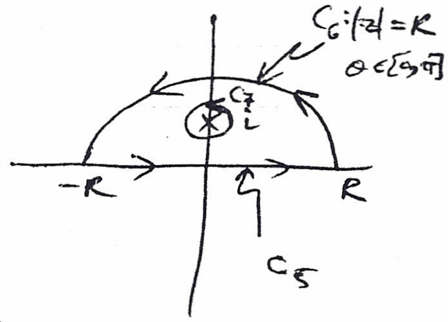
$$= \frac{\pi}{4}(-2+i\pi) - \lim_{R\to\infty}\left(\int_{1/R}^{R}\!\!{}_{C_2} \frac{\log x\,dx}{(x^2+1)^2} + i\pi\int_{1/R}^{R}\!\!{}_{C_2}\frac{dx}{(1+x^2)^2}\right) =$$

4 / 18.6

$$= \frac{\pi}{4}(-2+i\pi) - I - i\pi I' \quad \text{where } I' = \lim_{R\to\infty}\int_{1/R}^{R}\!\!{}_{C_2}\frac{dx}{(1+x^2)^2}$$

$$\therefore I = \frac{\pi}{4}(-2+i\pi) - I - i\pi I' \Rightarrow 2I = \frac{\pi}{4}(-2+i\pi) - i\pi I'$$

$$\therefore I = \frac{\pi}{8}(-2+i\pi) - \frac{i\pi}{2}I'$$



$$\because \int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^2} = 2\int_0^{\infty}\frac{dx}{(x^2+1)^2} = 2I'$$

$$\therefore 2I' = \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(x^2+1)^2} = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

$$\therefore f(x) = \frac{1}{(x^2+1)^2} \qquad \therefore f(z) \text{ has singularity at } z_0 = \pm i$$

$$\therefore 2I' = \lim_{R\to\infty}\int_{-R}^{R}\!\!{}_{C_5} f(z)\,dz = \lim_{R\to\infty}\left[\oint_{C_5+C_6} - \int_{C_6}\right] f(z)\,dz$$

$$\therefore \oint_{C_5+C_6} = \oint_{C_7} f(z)\,dz = \oint_{C_7}\frac{dz}{(z^2+1)^2} = \oint_{C_7}\frac{1/(z+i)^2}{(z-i)^2}\,dz = 2\pi i * \left(\frac{1}{(z+i)^2}\right)'\Bigg|_i =$$

$$= 2\pi i * \frac{-2}{(z+i)^3}\Bigg|_i = -4\pi i * \frac{1}{(2i)^3} = \frac{-4\pi i}{-8i} = \pi/2$$

$$\text{if } \left|\int_{C_6}\right| = \left|\int_{C_6} f(z)\,dz\right| \le \int_{C_6}\left|\frac{dz}{(z^2+1)^2}\right| \le \frac{\pi R}{\big||z|^2-1\big|^2_{|z|=R}} = \frac{\pi R}{(R^2-1)^2} = \frac{\pi/R^3}{(1-1/R^2)^2} = 0 \text{ as } R\to\infty$$

$$\therefore 2I' = \lim_{R\to\infty}\left[\oint_{C_5+C_6} - \int_{C_6} \to 0\right] f(z)\,dz = \pi/2 \qquad \therefore I' = \pi/4 \ (\therefore \text{OK } \boxed{4/18.6})$$

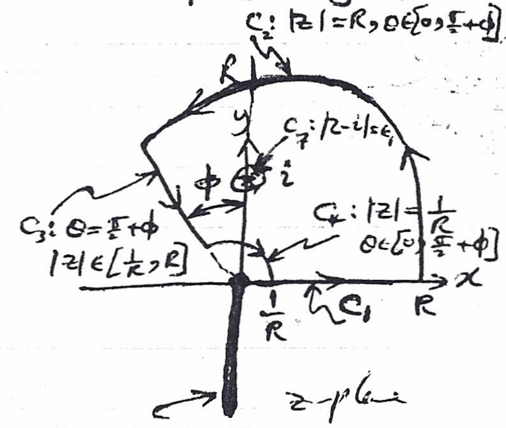
$$\therefore I = \frac{\pi}{8}(-2+i\pi) - \frac{i\pi}{2} * I' = -\frac{2\pi}{8} + \frac{i\pi^2}{8} - \frac{i\pi}{2}\cdot\frac{\pi}{4} = -\frac{\pi}{4} + \frac{i\pi^2}{8} - \frac{i\pi^2}{8} = -\pi/4$$

$$\therefore I = \int_0^{\infty}\frac{\log x}{(x^2+1)^2}\,dx = -\pi/4 \quad , \therefore \text{OK}$$

$\boxed{\dfrac{10b}{192}}$ $\quad I = \displaystyle\int_0^\infty \frac{x^a\,dx}{(x^2+1)^2}$ where $a \in (-1, 3)$, $x^a = \exp(a \operatorname{Log} x)$

$$= \lim_{R\to\infty} \int_{1/R}^{R} \frac{x^a\,dx}{(x^2+1)^2} = \lim_{R\to\infty} \int_{1/R}^{R} f(x)\,dx$$

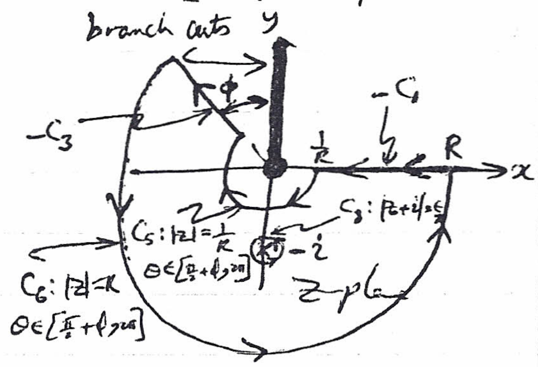
where $f(x) = \dfrac{x^a}{(x^2+1)^2}$



$\therefore\ f(z) = \dfrac{z^a}{(z^2+1)^2}$ has singularities at $z = \pm i$

where $z^a = \exp(a \log z)$

(Note: you can not use previous tricks since $f(z)$ is not even, so proceed as follows:)



Let $f_1(z) = \dfrac{\exp(a \log z)}{(z^2+1)^2}$: $\angle z \in (-\pi/2, \frac{3\pi}{2})$

& $f_2(z) = \dfrac{\exp(a \log z)}{(z^2+1)^2}$: $\angle z \in (\frac{\pi}{2}, \frac{5\pi}{2})$

Note that $f_1(z) = f_2(z)$ at $z: \angle z \in (-\pi/2, \frac{3\pi}{2}) \cap (\frac{\pi}{2}, \frac{5\pi}{2}) = (\frac{\pi}{2}, \frac{3\pi}{2})$

$$\therefore\ I = \lim_{R\to\infty} \int_{\frac{1}{R}\,C_1}^{R} f_1(z)\,dz = \lim_{R\to\infty} \left(\oint_{C_1+C_2+C_3+C_4} - \int_{C_2} - \int_{C_3} - \int_{C_4} \right) f_1(z)\,dz$$

$$= \lim_{R\to\infty} \left[\left(\oint_{C_7: |z-i| = \epsilon_1} - \int_{C_2} - \int_{C_4} \right) f_1(z)\,dz + \int_{-C_3} f_1(z)\,dz \right] =$$

$$= \lim_{R\to\infty} \left[\left(\oint_{C_7} - \int_{C_2} - \int_{C_4} \right) f_1(z)\,dz + \int_{-C_3} f_2(z)\,dz \right] =$$

$$= \lim_{R\to\infty} \left[\left(\oint_{C_7} - \int_{C_2} - \int_{C_4} \right) f_1(z)\,dz + \left(\oint_{-C_3+C_6-C_1+C_5} - \int_{C_6} - \int_{-C_1} - \int_{C_5} \right) f_2(z)\,dz \right]$$

$$= \lim_{R\to\infty} \left[\left(\oint_{C_7} - \int_{C_2} - \int_{C_4} \right) f_1(z)\,dz + \left(\oint_{C_8: |z+i| = \epsilon_2} - \int_{C_6} + \int_{C_1} - \int_{C_5} \right) f_2(z)\,dz \right] \quad (*)$$

$$\therefore\ \oint_{C_7} f_1(z)\,dz = \oint_{C_7} \frac{\exp(a \log z)}{(z^2+1)^2}\,dz = \oint_{C_7} \frac{e^{a \log z}/(z+i)^2}{(z-i)^2}\,dz = 2\pi i \left(\frac{e^{a \log z}}{(z+i)^2} \right)' \Bigg|_i =$$

$$= 2\pi i \left(\frac{e^{a \log z} \cdot (\frac{a}{z})(z+i)^2 - 2(z+i)\,e^{a \log z}}{(z+i)^4} \Bigg| \right) = \frac{2\pi i}{(2i)^4} \cdot e^{a \log i} \cdot \left(\frac{a}{i}(2i)^2 - 2(2i) \right) =$$

$$= \frac{2\pi i}{16} \cdot e^{a(\ln 1 + i\frac{\pi}{2})} \cdot (4ai - 4i) = \frac{\pi i}{8} * 4i * (a-1) \cdot e^{a\pi i/2} = -\frac{\pi}{2}(a-1)\, e^{a\pi i/2}$$

$$\left|\int_{C_2} f_1(z)\,dz\right| = \left|\int_{C_2} \frac{\exp(a\log z)}{(z^2+1)^2}\,dz\right| \le \int_{C_2} \frac{|e^{a\log z}|\,|dz|}{(|z^2|-1)^2} = \int_{C_2} \frac{|e^{a(\ln R + i\theta)}|\,|dz|}{(R^2-1)^2} =$$

$$= \frac{e^{a\ln R} \cdot 1 \cdot (\pi/2+\phi)R}{(R^2-1)^2} = \frac{(\frac{\pi}{2}+\phi)\frac{e^{\ln R^a}}{R^3}}{(1-1/R^2)^2} = (\text{as } R\to\infty)\,(\tfrac{\pi}{2}+\phi)\,\frac{R^a}{R^3} =$$

$$= (\tfrac{\pi}{2}+\phi)\frac{1}{R^{3-a}} = (\text{since } a<3)\,(\tfrac{\pi}{2}+\phi)\frac{1}{R^{+\epsilon}} = (\text{as } R\to\infty) = 0$$

$$\left|\int_{C_4} f_1(z)\,dz\right| = \left|\int_{C_4} \frac{\exp(a\log z)}{(z^2+1)^2}\,dz\right| \le \frac{\int |e^{a(\ln(\frac{1}{R})+i\theta)}|\,|dz|}{(|z^2|-1)^2} =$$

$$= \frac{e^{-a\ln R} \cdot (\frac{\pi}{2}+\phi)(\frac{1}{R})}{[(\frac{1}{R})^2-1]^2} = (\text{as } R\to\infty)\,(\tfrac{\pi}{2}+\phi)\frac{e^{\ln(R)^{-a}}}{R} = (\tfrac{\pi}{2}+\phi)\frac{R^{-a}}{R} =$$

$$= (\tfrac{\pi}{2}+\phi)\frac{1}{R^{1+a}} = (\text{since } a>-1)\,(\tfrac{\pi}{2}+\phi)\frac{1}{R^{+\epsilon}} = (\text{as } R\to\infty) = 0$$

$$\oint_{C_8} f_2(z)\,dz = \oint_{C_8} \frac{\exp(a\log z)}{(z^2+1)^2}\,dz = \oint_{C_8} \frac{e^{a\log z}/(z-i)^2}{(z+i)^2}\,dz =$$

$$= 2\pi i * \left(\frac{e^{a\log z}}{(z-i)^2}\right)'\Bigg|_{-i} = 2\pi i * \frac{e^{a\log z}(\frac{a}{z})(z-i)^2 - 2(z-i)e^{a\log z}}{(z-i)^4}\Bigg|_{-i} =$$

$$= \frac{2\pi i}{(-2i)^4} \cdot e^{a\log(-i)}\left[(\tfrac{a}{-i})(-2i)^2 - 2(-2i)\right] = \frac{2\pi i}{16} \cdot e^{a(\ln 1 + i3\pi/2)} \cdot (-4ai + 4i) =$$

$$= \frac{\pi i}{8}\, e^{3a\pi i/2} * 4i(1-a) = -\frac{\pi}{2}(1-a)\, e^{3\pi a i/2} = +\frac{\pi}{2}(a-1)\, e^{3\pi a i/2}$$

$$\left|\int_{C_6} f_2(z)\,dz\right| = \left|\int_{C_6} \frac{\exp(a\log z)\,dz}{(z^2+1)^2}\right| \le \int_{C_6} \frac{|e^{a(\ln R + i\theta)}|\,|dz|}{(|z^2|-1)^2} = \int_{C_6} \frac{|e^{a\ln R}|\,|dz|}{(R^2-1)^2} =$$

$$= \int_{C_6} \frac{e^{\ln R^a}|dz|}{(R^2-1)^2} = \int_{C_6} \frac{R^a|dz|}{(R^2-1)^2} = \frac{R^a(\frac{3\pi}{2}-\phi)R}{(R^2-1)^2} = \frac{(\frac{3\pi}{2}-\phi)R^a/R^3}{(1-1/R^2)^2} =$$

$$= (\text{as } R\to\infty)\,(\tfrac{3\pi}{2}-\phi)\cdot\frac{1}{R^{3-a}} = (\text{since } a<3)\,(\tfrac{3\pi}{2}-\phi)\frac{1}{R^{+\epsilon}} = 0 \text{ as } R\to\infty$$

$$\int_{C_1} f_2(z)\,dz = \int_{C_1} \frac{e^{a\log z}}{(z^2+1)^2}\,dz = \int_{\frac{1}{R}}^{R} \frac{e^{a(\ln x + 2\pi i)}}{(x^2+1)^2}\,dx = \int_{\frac{1}{R}}^{R} \frac{e^{a\ln x} \cdot e^{2\pi a i}}{(x^2+1)^2}\,dx$$

$$= \int_{\frac{1}{R}}^{R} \frac{e^{\ln x^a} \cdot e^{2\pi a i}}{(x^2+1)^2}\,dx = e^{2\pi a i}\int_{\frac{1}{R}}^{R} \frac{x^a\,dx}{(x^2+1)^2} = (\text{as } R\to\infty)\; e^{2\pi a i} * I$$

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$\neq \left| \int_{C_5} f_2(z)\,dz \right| = \left| \int_{C_5} \frac{e^{a\log z}}{(z^2+1)^2}\,dz \right| \leq \int_{C_5} \frac{\left|e^{a(\ln\frac{1}{R}+i\theta)}\right|}{(|z^2|-1)^2}|dz| =$

$$= \int_{C_5} \frac{e^{a\ln\frac{1}{R}}}{\left[\left(\frac{1}{R}\right)^2-1\right]^2}|dz| = \int_{C_5} \frac{e^{\ln\left(\frac{1}{R}\right)^a}}{\left[\left(\frac{1}{R}\right)^2-1\right]^2}|dz| = \frac{\left(\frac{1}{R}\right)^a\left(\frac{3\pi}{2}-\phi\right)\left(\frac{1}{R}\right)}{\left[\left(\frac{1}{R}\right)^2-1\right]^2} = (\text{as } R\to\infty)\left(\frac{3\pi}{2}-\phi\right)\frac{1}{R^{1+a}}$$

$$= (\text{since } a>-1)\left(\frac{3\pi}{2}-\phi\right)\frac{1}{R^{+\epsilon}} = 0 \quad \text{as } R\to\infty$$

$\therefore$ Back to $(*)$

$$\therefore I = \lim_{R\to\infty}\left[\left(\oint_{C_7} - \int_{C_2} - \int_{C_4}\right) f_1(z)\,dz + \left(\oint_{C_8} - \int_{C_6} + \int_{C_1} - \int_{C_5}\right) f_2(z)\,dz\right]$$

$$= -\frac{\pi}{2}(a-1)e^{a\pi i/2} + \frac{\pi}{2}(a-1)e^{3a\pi i/2} + e^{2\pi a i} * I$$

$$\therefore I\left(1-e^{2\pi a i}\right) = \frac{\pi}{2}(a-1)\left(e^{3a\pi i/2} - e^{a\pi i/2}\right) = \frac{\pi}{2}(a-1)e^{a\pi i/2}\left(e^{a\pi i}-1\right)$$

$$\therefore I = \frac{\pi}{2}\cdot(a-1)e^{a\pi i/2}\cdot\frac{\left(e^{a\pi i}-1\right)}{1-\left(e^{a\pi i}\right)^2} = \frac{\pi}{2}(1-a)\cdot e^{a\pi i/2}\cdot\frac{\left(1-e^{a\pi i}\right)}{\left(1-e^{a\pi i}\right)\left(1+e^{a\pi i}\right)} =$$

$$= \frac{\pi}{2}\cdot(1-a)\cdot e^{a\pi i/2}\cdot\frac{1}{1+e^{a\pi i}} = \frac{\pi(1-a)/2}{e^{-a\pi i/2}+e^{a\pi i/2}} = \frac{\pi(1-a)/2}{\left(\frac{e^{i(a\pi/2)}+e^{-i(a\pi/2)}}{2}\right)*2} =$$

$$= \frac{\pi(1-a)/2}{2\cos(a\pi/2)} = \frac{\pi(1-a)}{4\cos(a\pi/2)}$$

$$\therefore \int_0^\infty \frac{x^a\,dx}{(x^2+1)^2} = \frac{\pi(1-a)}{4\cos(a\pi/2)} \quad \neq \quad a\in(-1,3) \quad \therefore \text{OK}$$

$\frac{15}{194}$

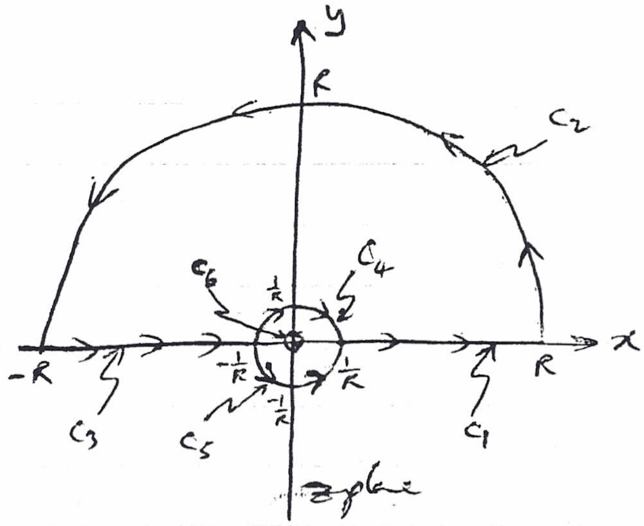
$I=\int_0^\infty \frac{\sin^2 x}{x^2}\,dx =$ (since the integrand is even) $\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,dx$

$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{\frac{1-\cos 2x}{2}}{x^2}\,dx = \frac{1}{4}\lim_{R\to\infty}\int_{-R}^{R}\frac{1-\operatorname{Re} e^{2ix}}{x^2}\,dx =$

$= \frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\int_{-R}^{R}\frac{1-e^{2ix}}{x^2}\,dx =$

$= \frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx,$

where $f(x)=\frac{1-e^{2ix}}{x^2}$



∴ $f(z)$ has singularity at $z_0=0$

∴ $I = \frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\int_{C_3+C_1} f(z)\,dz =$

$= \frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\int_{C_3+C_1}\frac{1-e^{2iz}}{z^2}\,dz = \frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\left(\int_{C_3+C_1}\frac{1}{z^2}\,dz - \int_{C_3+C_1}\frac{e^{2iz}}{z^2}\,dz\right)$

$= \left[\frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\left(\int_{C_3}\frac{dz}{z^2}+\int_{C_1}\frac{dz}{z^2}\right)\right] + I'$, where $I' = -\frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\int_{C_3+C_1} g(z)\,dz$, $g(z)=\frac{e^{2iz}}{z^2}$

$= \left[\frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\left(-\frac{1}{z}\Big|_{C_3} - \frac{1}{z}\Big|_{C_1}\right)\right] + I' = I' - \frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\left(\frac{1}{x}\Big|_{-R}^{-1/R} + \frac{1}{x}\Big|_{1/R}^{R}\right) =$

$= I' - \frac{1}{4} * \lim_{R\to\infty}\left(-R - \frac{1}{-R} + \frac{1}{R} - R\right) = I' - \frac{1}{4} * \lim_{R\to\infty}\left(\cancel{\frac{2}{R}}^{\,0} - 2R\right) =$

$= I' + \frac{1}{2}\lim_{R\to\infty} R \qquad (*)$

Now, $I' = -\frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\int_{C_3+C_1} g(z)\,dz = -\frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\left(\frac{\overbrace{\oint_{C_3+C_4+C_1+C_2}}^{0\ (g(z)\text{ analytic within it})} - \int_{C_4} - \int_{C_2} + \oint_{C_3+C_5+C_1+C_2} - \int_{C_5} - \int_{C_2}}{2}\right) g(z)\,dz$

$= +\frac{1}{4}\operatorname{Re}\lim_{R\to\infty}\left(\frac{\int_{C_4} + 2\int_{C_2} - \oint_{C_6} + \int_{C_5}}{2}\right) g(z)\,dz = \frac{1}{8}\operatorname{Re}\lim_{K\to\infty}\left(\int_{C_4} + 2\int_{C_2} - \oint_{C_6} + \int_{C_5}\right) g(z)\,dz$

$= \frac{1}{8}\operatorname{Re}\lim_{K\to\infty}\left(\int_{C_4+C_5} + 2\int_{C_2} - \oint_{C_6}\right) g(z)\,dz. \qquad (**)$ 118

∴ $\int_{C_4+C_5} g(z)\,dz = \int_{C_4+C_5}\frac{e^{2iz}}{z^2}\,dz =$ (by expanding e^{2iz} by Maclaurin) $\int_{C_4+C_5}\frac{\sum_{n=0}^{\infty}\frac{(2iz)^n}{n!}}{z^2}\,dz =$

$= \int_{C_4+C_5}\sum_{n=0}^{\infty}\frac{(2i)^n}{n!} z^{n-2}\,dz = \sum_{n=0}^{\infty}\frac{(2i)^n}{n!}\int_{C_4+C_5;\ z=\frac{1}{R}e^{i\theta}} z^{n-2}\,dz = \sum_{n=0}^{\infty}\frac{(2i)^n}{n!}\int_{C_4+C_5}\left(\frac{e^{i\theta}}{R}\right)^{n-2}\frac{ie^{i\theta}}{R}\,d\theta =$

$$= \sum_{n=0}^{\infty} \frac{(2i)^n}{n!}\cdot\frac{i}{R^{n-1}}\cdot\int_{C_4+C_5} e^{i(n-1)\theta}\,d\theta = \text{(by expanding term by term)}$$

$$= \underset{n=0}{iR\int_{C_4+C_5} e^{-i\theta}d\theta} - \underset{n=1}{2\int_{C_4+C_5} d\theta} + \sum_{n=2}^{\infty}\frac{(2i)^n}{n!}\cdot\frac{i}{R^{n-1}}\cdot\frac{e^{i(n-1)\theta}}{i(n-1)}\Bigg|_{C_4+C_5} =$$

$$= iR\cdot\frac{e^{-i\theta}}{(-i)}\Bigg|_{C_4+C_5} - 2\theta\Big|_{C_4+C_5} + \sum_{n=2}^{\infty}\frac{(2i)^n}{n!}\cdot\frac{e^{i(n-1)\theta}}{(n-1)R^{n-1}}\Bigg|_{C_4+C_5} =$$

$$= -Re^{-i\theta}\Big|_{C_4} - Re^{-i\theta}\Big|_{C_5} - 2\theta\Big|_{C_4} - 2\theta\Big|_{C_5} + \sum_{n=2}^{\infty}\frac{(2i)^n}{n!}\cdot\frac{1}{(n-1)R^{n-1}}\cdot\left(e^{i(n-1)\theta}\Big|_{C_4} + e^{i(n-1)\theta}\Big|_{C_5}\right) =$$

$$= -Re^{-i\theta}\Big|_{\pi}^{0} - Re^{-i\theta}\Big|_{\pi}^{2\pi} - 2\theta\Big|_{\pi}^{0} - 2\theta\Big|_{\pi}^{2\pi} + \sum_{n=2}^{\infty}\frac{(2i)^n}{n!(n-1)R^{n-1}}\cdot\left(e^{i(n-1)\theta}\Big|_{\pi}^{0} + e^{i(n-1)\theta}\Big|_{\pi}^{2\pi}\right) =$$

$$= -R(1+1) - R(1+1) - 2(\cancel{0-\pi}) - 2(\cancel{2\pi-\pi}) + \sum_{n=2}^{\infty}\frac{(2i)^n}{n!(n-1)R^{n-1}}\cdot\left(1-\cos(n-1)\pi + 1 - \cos(n-1)\pi\right) =$$

$$= -4R + \sum_{n=2}^{\infty}\frac{(2i)^n * 2(1-\cos(n-1)\pi)}{n!(n-1)R^{n-1}} = -4R + \sum_{\substack{n=2 \\ n\text{ even}\geq 2}}^{\infty}\frac{(2i)^n * 4}{n!(n-1)R^{n-1}} + \underset{n\text{ odd}\geq 2}{0}$$

$$= -4R - \underset{n=2}{\frac{8}{R}} + \underset{n=4}{\frac{8}{9R^3}} - \underset{n=6}{\frac{16}{225R^5}} + \cdots = (\text{as } R\to\infty)\ -4R$$

$$\left|\int_{C_2} g(z)dz\right| \leq \int_{C_2}\left|\frac{e^{2iz}}{z^2}\right|\cdot|dz| = \int_{C_2}\frac{|e^{2ix-2y}|}{|z|^2}\cdot|dz| = \int_{C_2}\frac{|e^{2ix}|\cdot|e^{-2y}|}{|z|^2}|dz| =$$

$$= \int\frac{e^{-2y}}{R^2}\cdot|dz| \leq \frac{1}{R^2} * \pi R = \frac{\pi}{R} = (\text{as } R\to\infty)\ 0$$

$$\oint_{C_6} g(z)dz = \oint_{C_6}\frac{e^{2iz}}{z^2}dz = \frac{(e^{2iz})'}{1!}\Bigg|_{z=0} * 2\pi i = e^{2iz} * 2i\Big|_0 * 2\pi i = 4\pi i^2 = -4\pi$$

∴ Back to (**)

$$\therefore\ I' = \frac{1}{8}\,\mathrm{Re}\lim_{R\to\infty}\left(\int_{C_4+C_5} + 2\int_{C_2}^{\nearrow 0} - \oint_{C_6}\right)g(z)dz = \frac{1}{8}\,\mathrm{Re}\lim_{R\to\infty}(-4R+0+4\pi)$$

$$= \frac{1}{8}\lim_{R\to\infty} 4(\pi-R) = \frac{1}{2}\lim_{R\to\infty}(\pi-R) = \frac{\pi}{2} - \lim_{R\to\infty}\frac{R}{2}$$

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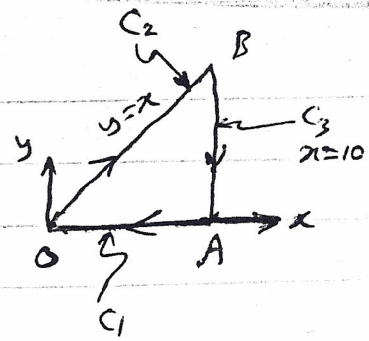
$$\therefore \text{ Back to } (*) \implies I = I' + \frac{1}{2}\lim_{R\to\infty} R = \frac{\pi}{2} - \lim_{R\to\infty}\frac{R}{2} + \lim_{R\to\infty}\frac{R}{2} =$$

$$= \frac{\pi}{2} + \lim_{R\to\infty}\left(-\frac{R}{2}+\frac{R}{2}\right) = \frac{\pi}{2} + \lim_{R\to\infty} 0 = \frac{\pi}{2} \qquad \therefore \int_0^{\infty}\frac{\sin^2 x}{x^2}dx = \frac{\pi}{2},\ \therefore \text{OK}$$

$\#$ $\displaystyle\oint_{\substack{C\\(CW)}} (z-\bar{z})\,dz = \left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right)(z-\bar{z})\,dz =$



$$= \left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right)\left[x+iy-(x-iy)\right](dx+i\,dy) =$$

$$= \left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right) 2iy\,(dx+i\,dy)$$

$\therefore$ $\displaystyle\int_{C_1:\ y=0\ \therefore dy=0,\ x:10\to 0} 2iy\,(dx+i\,dy) = \int_{10}^{0} 2i(0)(dx+i0) = 0$

$\&$ $\displaystyle\int_{C_2:\ y=x,\ dy=dx,\ x:0\to 10} 2iy\,(dx+i\,dy) = \int_{0}^{10} 2ix(1+i)\,dx = 2i(1+i)\frac{x^2}{2}\Big|_0^{10} = (i-1)(100-0)$

$\&$ $\displaystyle\int_{C_3:\ x=10,\ dx=0,\ y:10\to 0} 2iy\,(dx+i\,dy) = \int_{10}^{0} 2iy(0+i\,dy) = 2i\int_{10}^{0} iy\,dy = -2\frac{y^2}{2}\Big|_{10}^{0} = y^2\Big|_0^{10} = 100-0$

$\therefore$ $\displaystyle\oint_C (z-\bar{z})\,dz = 0 + 100(i-1) + 100 = 100i$

$\#$ $\displaystyle\oint_{\substack{C\\(CW)}} \frac{dz}{z^2-8z+25} = I$, singularity at $z = 4 \pm i3$

$\therefore$ Only $4+3i$ is within C

$\therefore$ $\displaystyle I = -\oint_{\substack{C\\(CCW)}} \frac{dz}{z^2-8z+25} = -2\pi i \operatorname{Res}\left(\frac{1}{z^2-8z+25}\right)\Bigg|_{4+3i} =$

$$= -2\pi i\left(\frac{z-(4+3i)}{z^2-8z+25}\right)\Bigg|_{4+3i} = -2\pi i\left(\frac{1}{2z-8}\right)\Bigg|_{4+3i} = \frac{-2\pi i}{8+6i-8} =$$

$$= \frac{-2\pi i}{6i} = -\pi/3$$

$\therefore$ $\displaystyle\oint_{\substack{C\\(CW)}} \frac{dz}{z^2-8z+25} = -\pi/3 .$

$\boxed{\frac{1}{200}}$ $w = f(z) = z^2$, $z_0 = 2+i$

$\therefore f'(z_0) = 2z_0 \neq 0$ $\therefore$ mapping is conformal

Magnification $= |f'(z_0)| = |2z_0| = 2|2+i| = 2\sqrt{5}$

& Rotation $= \angle f'(z_0) = \angle 2z_0 = \angle z_0 = \tan^{-1}\frac{1}{2} = 26.6^\circ$

$\boxed{\frac{2}{200}}$ $w = \frac{1}{z} = f(z)$ $\therefore f'(z) = \frac{-1}{z^2}$

$\therefore$ angle of rotation $= \arg f'(z)$

ⓐ at $z=1$ $\therefore f'(z) = \frac{-1}{1^2} = -1$ $\therefore \angle f'(z) = \pi$

$\therefore$ Angle of rotation is π.

ⓑ at $z=i$ $\therefore f'(z) = \frac{-1}{i^2} = \frac{-1}{-1} = 1$ $\therefore \angle f'(z) = 0$

$\therefore$ No rotation.

$\boxed{\frac{4}{200}}$ $w = f(z) = z^n$ $\therefore f'(z) = nz^{n-1}$

$\therefore f'(z_0) = nz_0^{n-1} = n(r_0 e^{i\theta_0})^{n-1} = nr_0^{n-1} \cdot e^{i(n-1)\theta_0}$

$\therefore$ Rotation is $\angle f'(z_0) = (n-1)\theta_0$

& Magnification $= |f'(z_0)| = n \cdot r_0^{n-1}$

$\frac{5}{200}$

$w = f(z) = e^{z}$

To prove conformity of mapping $\therefore$ f must be analytic with $f'(z) \neq 0$

$\because f(z) = e^{z}$ $\therefore$ $f(z)$ is analytic everywhere

$\because f'(z) = e^{z} = e^{x+iy} = e^{x} \cdot e^{iy}$

$\therefore |f'(z)| = e^{x} \neq 0$ at all

$\therefore$ e^{z} maps conformally <u>everywhere</u> in the z plane.

$\frac{5}{200}$

$w = f(z) = \sin z$

$\therefore f'(z) = \cos z = 0$ at $z = \text{odd } \pi/2 = (2n+1)\frac{\pi}{2}$

$\therefore$ Mapping is conformal at <u>all</u> z except $(2n+1)\pi/2$

$\frac{\text{I}}{207}$ $u(x,y) = x^3 - 3xy^2$ Let the harmonic conjugate be v

$\therefore v_y = u_x = 3x^2 - 3y^2$

$\therefore v = \int (3x^2 - 3y^2)\partial y = 3x^2y - y^3 + g(x)$

$\therefore v_x = 6xy + g'(x)$

but $v_x = -u_y = -(-6xy) = 6xy$

$\therefore 6xy = 6xy + g'(x)$ $\quad \therefore g'(x) = 0$ $\quad \therefore g(x) = c$

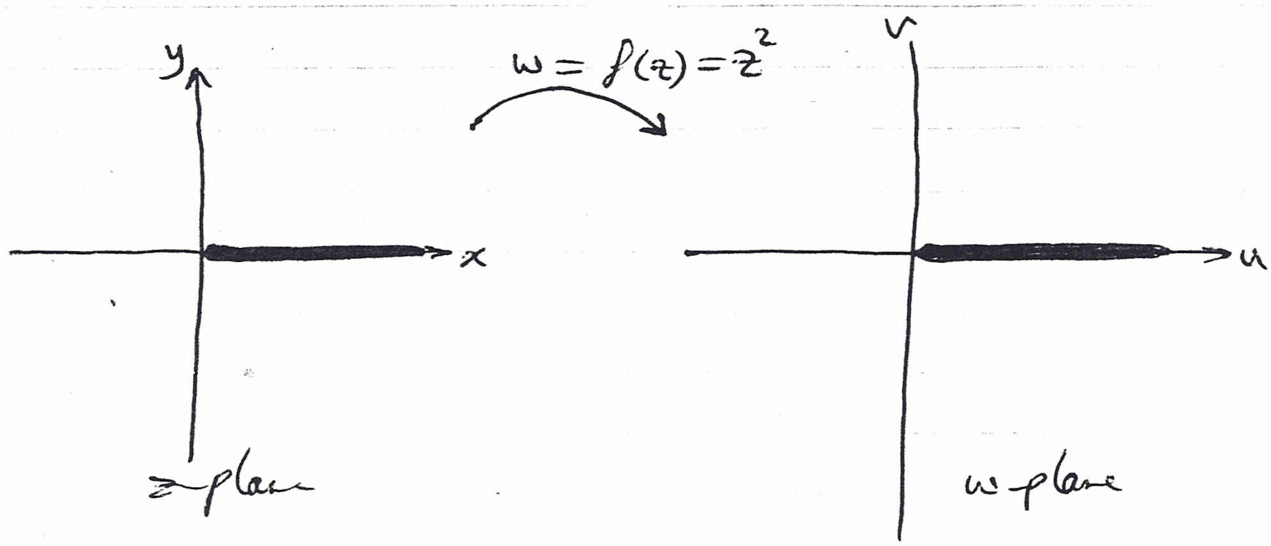
$\therefore v = 3x^2y - y^3 + c$

$\therefore f(z) = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3 + c)$

$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + ic = (x+iy)^3 + ic$

$\therefore f(z) = z^3 + A$ where A & c are complex constants.

$\dfrac{4}{207}$



$\because\ w = z^2 = f(z)\ \therefore\ f'(z) = 2z$ Conformal in the positive x-axis, $x \neq 0$ (bold lines)

$\because\ h(u,v) = e^{-u}\cos v$ is harmonic $\therefore$ So is $h(x,y)$

$\therefore\ \nabla h(u,v) = \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)h(u,v) = h_u + i h_v =$

$= -e^{-u}\cos v - i e^{-u}\sin v$

$\therefore\ \nabla h(u,0) = -e^{-u}\cos 0 - i e^{-u}\sin 0 = -e^{-u}$

$\therefore$ The gradient of h along $+u$ is pure real $\therefore$

$\therefore$ The normal derivative of h along $+u$ is zero (i.e ∇h along $+u$ is pure real)

$\therefore$ By theorem pp. 206 the normal derivative of h along $+x$ is zero.

To prove this:

$\therefore\ \nabla h(x,y) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)h(x,y) = h_x + i h_y = \left(-e^{-u}\cos v.\ u_x - e^{-u}\sin v.\ v_x\right)$

$+ i\left(-e^{-u}.\cos v.\ u_y - e^{-u}\sin v.\ v_y\right)$

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$\because\ w = u + iv = z^2 = (x+iy)^2 = x^2 - y^2 + i\,2xy$

$\therefore\ u = x^2 - y^2$, $u_x = 2x$, $u_y = -2y$ and at the x-axis $u_{y=0}$

$\&\ v = 2xy$, $v_x = 2y$, $v_y = 2x$ " " " " $: v = v_x = 0$

$\therefore\ \nabla h(x,y)\Big|_{\text{at } x\text{-axis}} = -e^{-u}\cos 0 . 2x - e^{-u}\cancel{\sin 0}.0 + i(-e^{-u}\cancel{\cos 0}.0 - e^{-u}\cancel{\sin 0}.2x) = -2xe^{-x^2}$

$\therefore$ The gradient of h along $+x$ is pure real, i.e. the normal derivative of

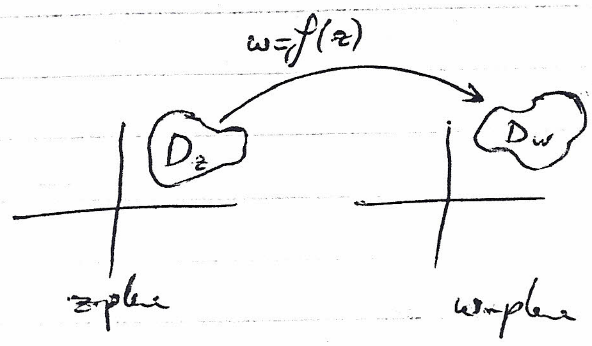
$\boxed{\frac{7}{208}}$

$h(u,v)$ is harmonic

$\therefore h_{uu} + h_{vv} = 0$

Prove that:

$h_{xx} + h_{yy} = (h_{uu} + h_{vv}) |f'(z)|^2$



$h(u,v) = h\big(u(x,y), v(x,y)\big)$

where $u + iv = w = f(z)$ (analytic) & $u_x = v_y$ & $u_y = -v_x$

$\therefore h_x = h_u \cdot u_x + h_v \cdot v_x$ & $h_y = h_u \cdot u_y + h_v \cdot v_y$

$\therefore h_{xx} = (h_{uu} \cdot u_x + h_{uv} \cdot v_x) u_x + h_u u_{xx} + (h_{vu} \cdot u_x + h_{vv} \cdot v_x) v_x + h_v \cdot v_{xx}$

& $h_{yy} = (h_{uu} \cdot u_y + h_{uv} \cdot v_y) u_y + h_u \cdot u_{yy} + (h_{vu} \cdot u_y + h_{vv} \cdot v_y) v_y + h_v \cdot v_{yy}$

$\therefore h_{xx} + h_{yy} = h_{uu} [u_x^2 + u_y^2] + h_{vv} [v_x^2 + v_y^2] + 2h_{uv} [v_x u_x + v_y u_y]$

$+ h_u [u_{xx} + u_{yy}] + h_v [v_{xx} + v_{yy}]$

$\because f$ is analytic $\therefore u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$ (both harmonic)

moreover $u_y = -v_x$, $v_y = u_x$

$\therefore h_{xx} + h_{yy} = h_{uu} \cdot [u_x^2 + v_x^2] + h_{vv} \cdot [v_x^2 + u_x^2] + 2h_{uv} [\underbrace{\cancel{v_x u_x + u_x(-v_x)}}_{0}] =$

$= (h_{uu} + h_{vv})(u_x^2 + v_x^2)$

but: $f'(z) = \frac{df}{dz} = \frac{d(u+iv)}{d(x+iy)} =$ (for y constant) $u_x + iv_x =$ (for x constant) $\frac{u_y + iv_y}{i}$

$\therefore |f'(z)|^2 = u_x^2 + v_x^2$

$\therefore h_{xx} + h_{yy} = (h_{uu} + h_{vv}) |f'(z)|^2$ $\therefore$ OK

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$\because h_{uu} + h_{vv} = 0$ because h is harmonic in D_w

$\therefore h_{xx} + h_{yy} = 0$ i.e. h is harmonic in D_z $\therefore$ OK

$\boxed{\frac{2}{224}}$

$$w = f(z) = i\frac{1-z}{1+z} = i\frac{1-z-1+1}{1+z} = i\cdot\frac{2-(1+z)}{1+z} = i\left(\frac{2}{1+z}-1\right)$$

$= i\left(\frac{2}{w_1}-1\right)$ where $w_1 = 1+z$

$= i\left(2w_2 - 1\right)$ where $w_2 = \frac{1}{w_1}$

$= i(w_3 - 1)$ where $w_3 = 2w_2$

$= i\,w_4$ where $w_4 = w_3 - 1$

$\therefore\ w = i\,w_4$.

The problem transforms as shown here: where the upper half circle goes to the first quadrant and the segment AC goes to A'C'.
The values of V along corresponding segments are preserved.

$\because\ V_{xx} + V_{yy} = 0$

$\therefore\ V_{uu} + V_{vv} = 0$

$\therefore$ Simplest function to satisfy this requirement and boundary requirements is $V(u,v) = \frac{2}{\pi} * \angle w =$

$= \frac{2}{\pi}\tan^{-1}\frac{v}{u}$

(check: $V_{uu} + V_{vv} =$

$\frac{2}{\pi}\left(\frac{-\frac{v}{u^2}}{1+\left(\frac{v}{u}\right)^2}\right)_u + \frac{2}{\pi}\left(\frac{\frac{1}{u}}{1+\left(\frac{v}{u}\right)^2}\right)_v =$

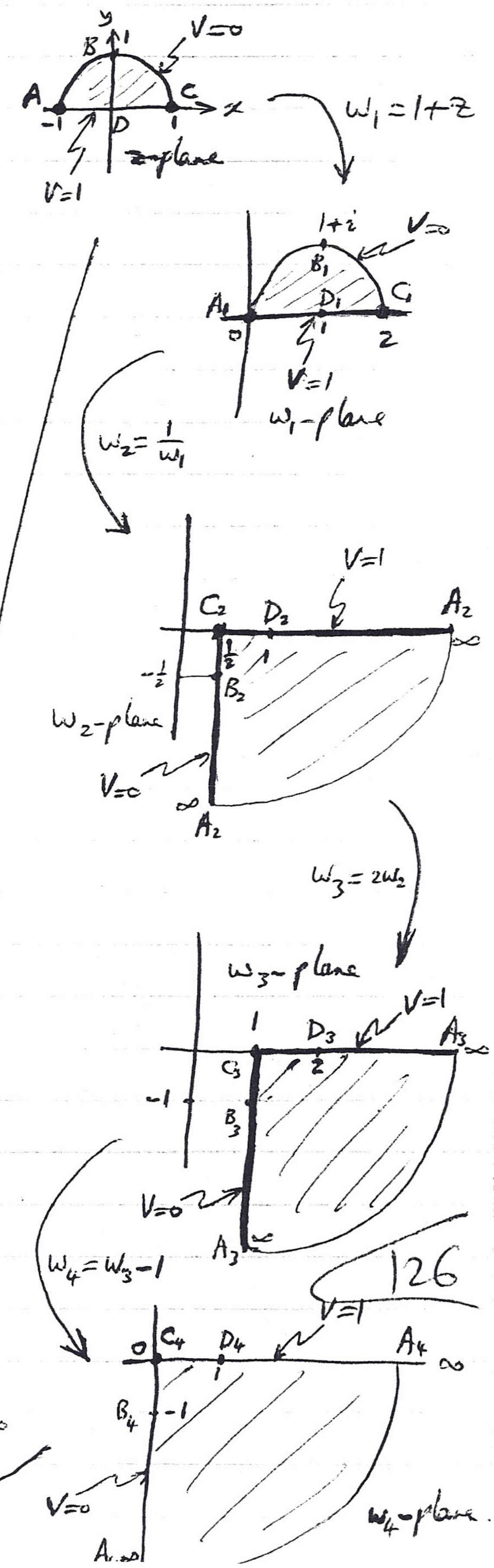
$= \frac{2}{\pi}\left[\frac{\frac{2v}{u^3}\left(1+\frac{v^2}{u^2}\right) + \frac{v}{u^2}\left(\frac{2v^2}{u^3}\right) - \frac{2v}{u^3}}{\left[1+\left(\frac{v}{u}\right)^2\right]^2}\right]$

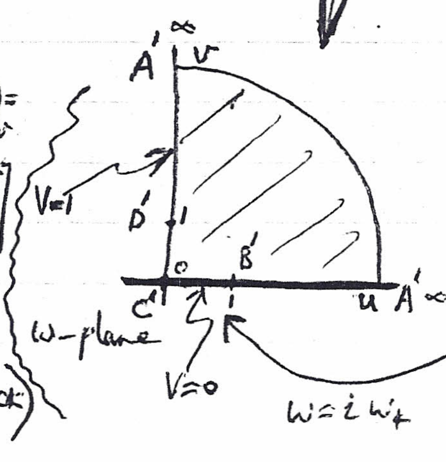
$= 0 \therefore$ harmonic $\therefore$ OK

& $V(u,0) = 0 \quad \therefore$ OK

& $V(0,v) = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1 \therefore$ OK)

$w = f(z)$





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$\therefore$ $V(u,v) = \frac{2}{\pi} \tan^{-1}\frac{v}{u}$ is the solution in the w-plane

$\because$ $w = f(z) = i \cdot \frac{1-z}{1+z}$

$\therefore$ $\tan^{-1}\frac{v}{u} = \angle w = \angle i + \angle 1-z - \angle 1+z = \frac{\pi}{2} + \tan^{-1}\frac{-y}{1-x} - \tan^{-1}\frac{y}{1+x}$

$\therefore$ $\frac{v}{u} = \tan\left(\frac{\pi}{2} + \tan^{-1}\frac{-y}{1-x} - \tan^{-1}\frac{y}{1+x}\right) = -\cot\left(\tan^{-1}\frac{-y}{1-x} - \tan^{-1}\frac{y}{1+x}\right)$

$\therefore$ $\frac{-u}{v} = \tan\left(\tan^{-1}\frac{-y}{1-x} - \tan^{-1}\frac{y}{1+x}\right) = \frac{\frac{-y}{1-x} - \frac{y}{1+x}}{1+\left(\frac{-y}{1-x}\right)\left(\frac{y}{1+x}\right)} =$

$= \frac{-y(1+\cancel{x}) - y(1-\cancel{x})}{1-x^2-y^2} = \frac{-2y}{1-x^2-y^2}$

$\therefore$ $\frac{u}{v} = \frac{2y}{1-x^2-y^2}$ $\qquad$ $\therefore$ $\frac{v}{u} = \frac{1-x^2-y^2}{2y}$

$\because$ $V(u,v) = \frac{2}{\pi} \tan^{-1}\frac{v}{u}$

$\therefore$ $V(x,y) = \frac{2}{\pi} \tan^{-1}\left(\frac{1-x^2-y^2}{2y}\right)$

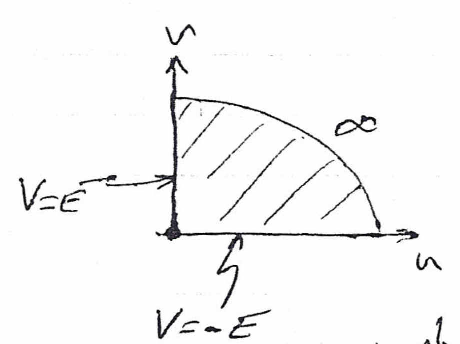
$\therefore$ The solution in the z-plane is:

$$V(x,y) = \frac{2}{\pi} \tan^{-1}\left(\frac{1-x^2-y^2}{2y}\right)$$

\#

$\therefore$ As in $\boxed{\frac{2}{224}}$ $\therefore$ $V(u,v) = \frac{4E}{\pi} \tan^{-1}\frac{v}{u} - E$

$\therefore$ $V(u,v) = E\left(\frac{4\tan^{-1}(v/u)}{\pi} - 1\right)$



$\because$ $\frac{v}{u} = \frac{1-x^2-y^2}{2y}$

$\therefore$ $V(x,y) = \frac{4E}{\pi}\left[\tan^{-1}\left(\frac{1-x^2-y^2}{2y}\right) - \frac{\pi}{4}\right]$

$\frac{7}{226}$

First we want to transform the problem to a simpler space where the segments $V=1$ & $V=0$ are joined together.

Consider the transformation $w_1 = e^z$ where the problem is mapped as follows:

$y=0$ (x-axis) goes to $e^{x+i0} = e^x$

$\therefore (0,0) \to (1,0)$, $(\infty,0) \to (\infty,0)$, $(-\infty, 0) \to (0,0)$

$y=\pi$ (other boundary) goes to $e^{x+i\pi} = -e^x$

$\therefore (\infty,\pi) \to (-\infty,0)$, $(0,\pi) \to (-1,0)$, $(-\infty,\pi) \to (0,0)$

Hence, the problem is simplified a bit to have the shape of that problem given in 5 pp. 225 with $a=1$.

This can then be solved using the transformation
$w = \log\left(\frac{w_1-1}{w_1+1}\right)$, $w_1 \neq 1$ & $\angle w_1 \pm 1 \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

where $\mathrm{Re}\, w = \ln\left|\frac{w_1-1}{w_1+1}\right|$ & $\mathrm{Im}\, w = \angle w_1 - 1 - \angle w_1 + 1$

Hence, at segment C_1D_1, $|w_1 - 1| < |w_1+1|$

$\therefore \mathrm{Re}\, w = \ln\left|\frac{w_1-1}{w_1+1}\right| < 0$ and (as $w_1 \to \infty$) $= 0$

& $\angle w_1 - 1 = \angle w_1 + 1 = 0$

$\therefore \mathrm{Im}\, w = 0$ and hence $C'D'$ is obtained.

& At segment F_1E_1, $|w_1-1| > |w_1+1|$

$\therefore \mathrm{Re}\, w = \ln\left|\frac{w_1-1}{w_1+1}\right| > 0$ and (as $w_1 \to -\infty$) $= 0$

whereas $\angle w_1 - 1 = \angle w_1 + 1 = \pi$

$\therefore \mathrm{Im}\, w = \pi - \pi = 0$ and hence $F'E'$ is obtained.

& At segment B_1A_1, $|w_1 - 1| < |w_1+1|$

$\therefore \mathrm{Re}\, w = \ln\left|\frac{w_1-1}{w_1+1}\right| < 0$ and (as $w_1 \to 0$) $= 0$

whereas $\angle w_1 - 1 = \pi$, $\angle w_1 + 1 = 0$

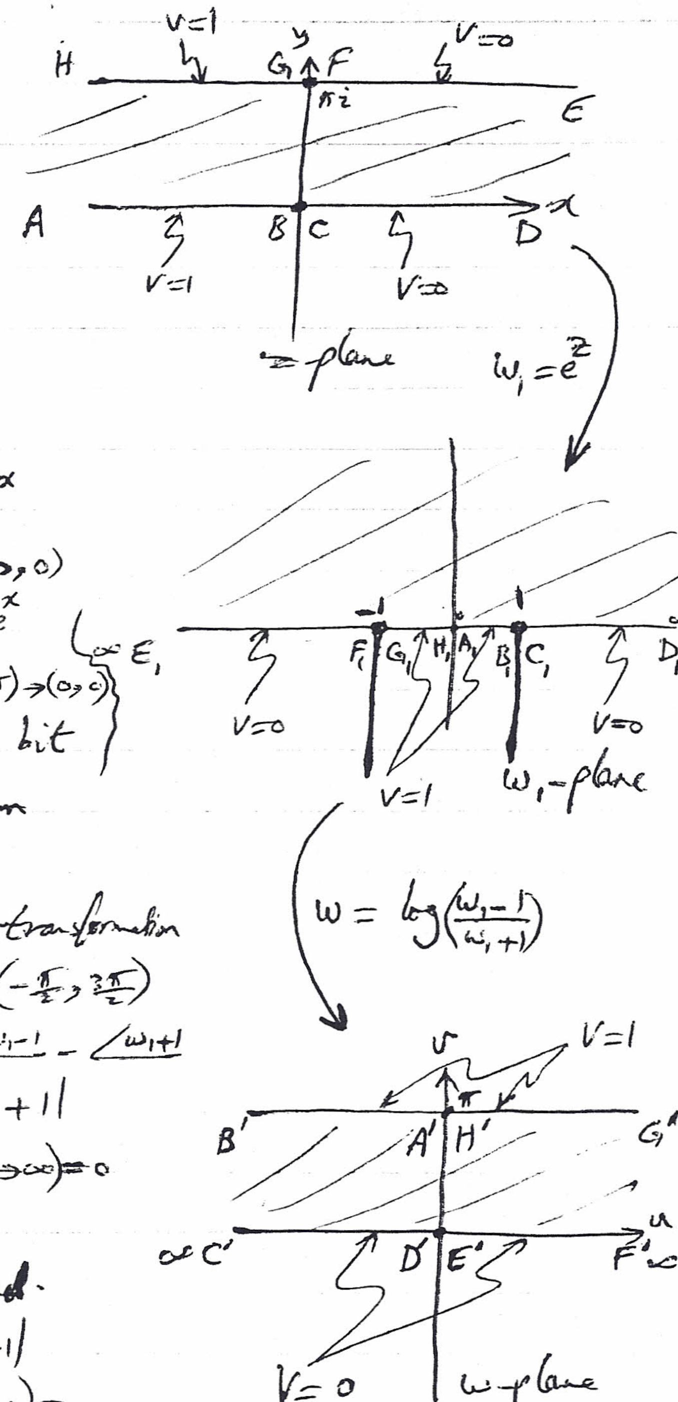
$\therefore \mathrm{Im}\, w = \pi - 0 = \pi$ and hence $B'A'$ is obtained.

Finally, at segment G_1H_1, $|w_1 - 1| > |w_1+1|$

$\therefore \mathrm{Re}\, w = \ln\left|\frac{w_1-1}{w_1+1}\right| > 0$ and (as $w_1 \to 0$) $= 0$

whereas $\angle w_1 - 1 = \pi$, $\angle w_1 + 1 = 0$

$\therefore \mathrm{Im}\, w = \pi - 0 = \pi$ and hence $G'H'$ is obtained.



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The problem is now very simple with $v=0$ held at $V=0$ and $v=\pi$ held at $V=1$

$\therefore V(u,v) = \frac{v}{\pi}$ is the solution.

(check: $V_{uu}+V_{vv} = 0+0=0 \quad \therefore OK$

$V(u,0)=0 \quad \therefore OK$ & $V(u,\pi) = \frac{\pi}{\pi} = 1 \quad \therefore OK$)

$$\because v = Im\, w = \angle w_1 - 1 - \angle w_1 + 1 = \tan^{-1}\frac{v_1}{u_1-1} - \tan^{-1}\frac{v_1}{u_1+1}$$

$$\therefore \tan v = \frac{\frac{v_1}{u_1-1} - \frac{v_1}{u_1+1}}{1+\frac{v_1}{u_1-1}\cdot\frac{v_1}{u_1+1}} = \frac{v_1(u_1+1)-v_1(u_1-1)}{u_1^2-1+v_1^2} = \frac{2v_1}{u_1^2+v_1^2-1}$$

but $w_1 = e^z = e^{x+iy} = e^x\cdot(\cos y + i\sin y) = u_1 + i v_1$

$\therefore u_1^2+v_1^2 = |w_1|^2 = e^{2x}$

& $v_1 = e^x \sin y$

$$\therefore \tan v = \frac{2v_1}{u_1^2+v_1^2-1} = \frac{2e^x\sin y}{e^{2x}-1} = \frac{\sin y}{\frac{e^x-e^{-x}}{2}} = \frac{\sin y}{\sinh x}$$

$$\therefore v = \tan^{-1}\frac{\sin y}{\sinh x}$$

$\therefore$ The solution in the z plane is:

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$$V(x,y) = \frac{1}{\pi}\cdot\tan^{-1}\left(\frac{\sin y}{\sinh x}\right).$$

check:

$V(x>0,0) = \frac{1}{\pi}\tan^{-1}\left(\frac{\sin 0}{\sinh +ve}\right) = \frac{1}{\pi}\tan^{-1}0^+ = 0 \quad \therefore OK$

$V(x>0,\pi) = \frac{1}{\pi}\tan^{-1}\left(\frac{\sin\pi}{\sinh +ve}\right) = \frac{1}{\pi}\tan^{-1}0^+ = 0 \quad \therefore OK$

$V(x<0,0) = \frac{1}{\pi}\tan^{-1}\left(\frac{\sin 0}{\sinh -ve}\right) = \frac{1}{\pi}\tan^{-1}0^- = \frac{1}{\pi}\cdot\pi = 1 \quad \therefore OK$

$V(x<0,\pi) = \frac{1}{\pi}\tan^{-1}\left(\frac{\sin\pi}{\sinh -ve}\right) = \frac{1}{\pi}\tan^{-1}0^- = \frac{1}{\pi}\cdot\pi = 1 \quad \therefore OK$

Let $w = f(z) = i \cdot \dfrac{R-z}{R+z} =$

$$= i \cdot \frac{2R-(R+z)}{R+z} = i\left(\frac{2R}{R+z} - 1\right)$$

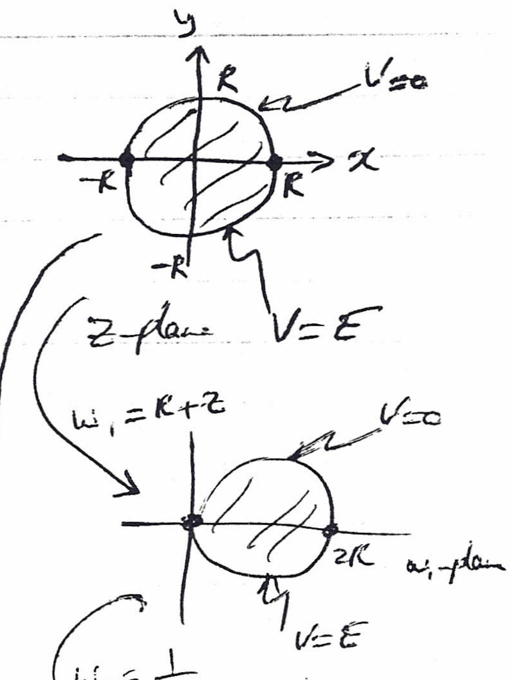
$$= \left(\text{as in exercise } \frac{2}{224}\right) i\left(\frac{2R}{w_1} - 1\right) \quad \text{where } w_1 = R+z$$

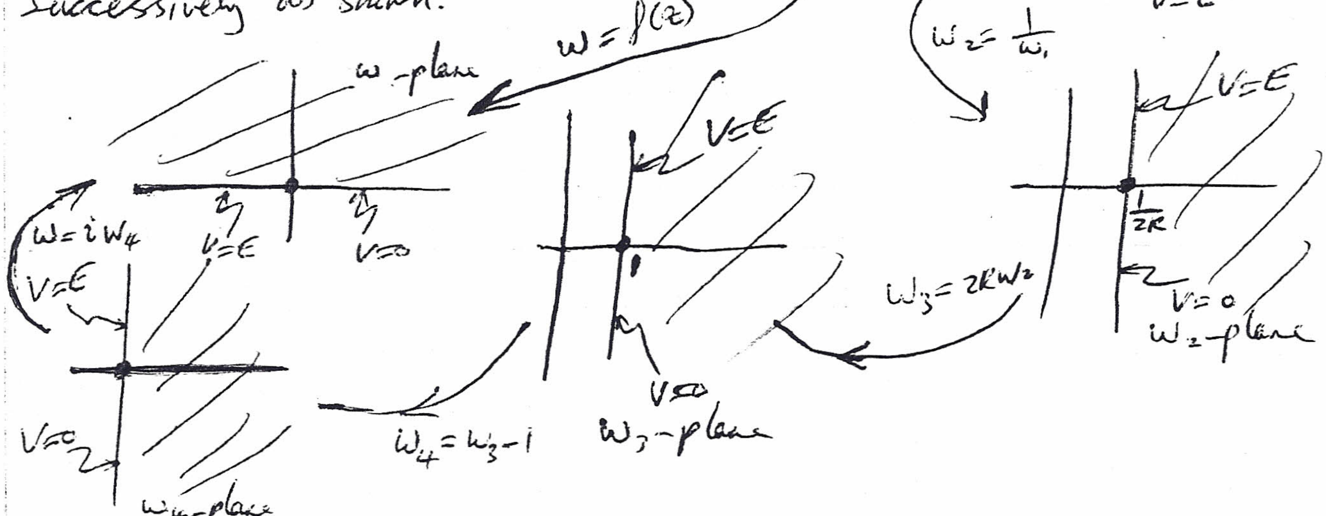
$$= i(2Rw_2 - 1) \qquad \text{where } w_2 = \frac{1}{w_1}$$

$$= i(w_3 - 1) \qquad \text{where } w_3 = 2Rw_2$$

$$= i\,w_4 \qquad \text{where } w_4 = w_3 - 1$$

$\therefore$ $w = i\,w_4$ and the problem transforms successively as shown.





$\therefore$ The solution is simply:

$$V(u,v) = \frac{E}{\pi}\angle w = \frac{E}{\pi}\tan^{-1}\frac{v}{u}$$

but $\angle w = \tan^{-1}\dfrac{v}{u} = \angle f(z) = \angle i + \angle R-z - \angle R+z =$

$$= \frac{\pi}{2} + \tan^{-1}\frac{-y}{R-x} - \tan^{-1}\frac{y}{R+x}$$

$$\therefore \tan\left(\tan^{-1}\frac{v}{u}\right) = \tan\left(\frac{\pi}{2} + \tan^{-1}\frac{-y}{R-x} - \tan^{-1}\frac{y}{R+x}\right)$$

$$\therefore \frac{v}{u} = -\cot\left(\tan^{-1}\frac{-y}{R-x} - \tan^{-1}\frac{y}{R+x}\right)$$

$$\therefore -\frac{u}{v} = \tan\left(\tan^{-1}\frac{-y}{R-x} - \tan^{-1}\frac{y}{R+x}\right) = \frac{\frac{-y}{R-x} - \frac{y}{R+x}}{1 + \left(\frac{-y}{R-x}\right)\left(\frac{y}{R+x}\right)} =$$

$$= \frac{-y(R+\cancel{x}) - y(R-\cancel{x})}{R^2 - x^2 - y^2} = \frac{-2yR}{R^2 - x^2 - y^2}$$

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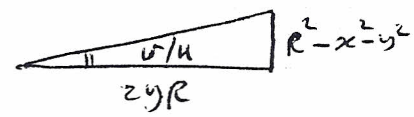
$$\therefore \frac{v}{u} = \frac{R^2 - x^2 - y^2}{2yR}$$

$\therefore$ The solution in the z-plane is

$$V(x,y) = \frac{E}{\pi} \tan^{-1}\left(\frac{R^2 - x^2 - y^2}{2yR}\right)$$

(Check:



At the boundary $x^2+y^2=R^2$ $\therefore V(x,y)$ upper half $= \frac{E}{\pi} \tan^{-1} \frac{0}{+ve} = 0$ $\therefore$ OK

& $V(x,y)$ lower half $= \frac{E}{\pi} \tan^{-1} \frac{0}{-ve} = \frac{E}{\pi} \cdot \pi = E$ $\therefore$ OK)

BIE